WORKING PAPER

Disproportionality at Fair Elections Cannot Exceed $\frac{1}{\sqrt{2}}$

A Note on the Gallagher Index and a New Normalised Measure

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This version: September 16, 2025

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ABSTRACT The Gallagher index of disproportionality is perhaps the most popular measure of electoral distortion in all of political science. While many assume that the index ranges from 0 (perfect proportionality) to 1 (perfect disproportionality), I prove that in any free and fair election its maximum possible value is bounded above by $\sqrt{\frac{N-1}{2N}}$, reaching a maximum of $\frac{1}{\sqrt{2}}$ where the number of parties, N, tends to infinity. This bound arises in the limiting case where all parties receive equal vote shares but only one secures representation. Building on this result, I then propose a new normalised version of the index that rescales disproportionality to lie between 0 and 1 under democratic conditions.

1 INTRODUCTION

The Gallagher index (1991) is perhaps the most popular measure of disproportionality in political science. Indeed, recent studies have used it to investigate topics as varied as intra-party cohesion (Matakos et al. 2024), subjective health and well-being (Toshkov and Mazepus 2023), and corporate investment decisions (Amore and Corina 2021). What's more, its impact has transcended academic research. After then-Liberal leader Justin Trudeau committed to making 2015 Canada's "last federal election conducted under the first-past-the-post voting system" (Liberal Party of Canada 2016, 8), the Gallagher index informed much of the work of the Canadian House of Commons Special Committee on Electoral Reform (2016).

Though the Gallagher index can, in principle, vary between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*), I prove in this paper that, at any free and fair election, the upper bound on the index is in fact $\sqrt{\frac{N-1}{2N}}$, where N is the number of parties. As such, in the absence of any interference, the largest value that the Gallagher index may take is $\frac{1}{\sqrt{2}}$, where the number of parties tends to ∞. In the sections that follows, I prove that the Gallagher index cannot

https://doi.org/10.33774/apsa-2025-khfwb ORCID: https://orcid.org/0000-0001-8517-5018 Content not peer-reviewed by APSA.

exceed $\frac{1}{\sqrt{2}}$ under democratic conditions, introduce a new normalised version of the index that varies between 0 and 1 at fair elections, and discuss how we might use the upper bound that I identify to ensure that our data are not prone to errors.

2 PROOF THAT DISPROPORTIONALITY CANNOT EXCEED $\frac{1}{\sqrt{2}}$

Let N be the number of parties, v be a vector of vote shares, and s be a vector of seat shares. Given this, we can compute Gallagher's (1991) index of disproportionality, D, as follows:

$$D = \sqrt{\frac{1}{2}\sum_{i=1}^{N}(v_i - s_i)^2} \tag{1}$$

The index takes any value between 0 (perfect proportionality) and 1 (perfect disproportionality). To be perfectly *proportional*, vote and seat shares must be identical. For instance, two parties might receive 80% and 20% of all votes and 80% and 20% of all seats. Likewise, to be perfectly *disproportional*, one party must win all of the seats while others win all of the votes. However, this outcome is undemocratic, since electoral systems *majorize* votes (Marshall, Olkin, and Pukelsheim 2002): they allocate seats to parties according to their support.

As such, the upper bound on the level of index must be lower than 1 at free and fair elections. Under such conditions, the most *disproportional* outcome that we could expect occurs where all parties receive *exactly* the same share of the vote, but where only one party wins any seats. That is, where all $v_i = \frac{1}{N}$ while $s_1 = 1$ and all $s_{i \neq 1} = 0$. Here, the sum in Equation 1 simplifies, providing us with a new upper bound on the index.

First, rewrite the sum in Equation 1 to separate $i = 1$ from all other cases:

$$\sum_{i=1}^{N}(v_i - s_i)^2 = (v_1 - s_1)^2 + \sum_{i=2}^{N}(v_i - s_i)^2 \tag{2}$$

Then, substitute in $v_i = \frac{1}{N}$, $s_1 = 1$, and $s_{i \neq 1} = 0$, giving:

$$\left(\frac{1}{N} - 1\right)^2 + (N-1)\left(\frac{1}{N} - 0\right)^2 \tag{3}$$

Equation 3 then simplifies as follows:

$$\left(\frac{1}{N} - 1\right)^2 + (N-1)\left(\frac{1}{N} - 0\right)^2 = \frac{(N-1)^2}{N^2} + \frac{N-1}{N^2} = \frac{N-1}{N} \tag{4}$$

And, substituting Equation 4 for the sum in Equation 1, we get:

$$D_{max} = \sqrt{\frac{1}{2} \times \frac{N-1}{N}} \tag{5}$$

Which again simplifies to give the upper bound on D for N vote-winning parties:

$$D_{max} = \sqrt{\frac{N-1}{2N}} \tag{6}$$

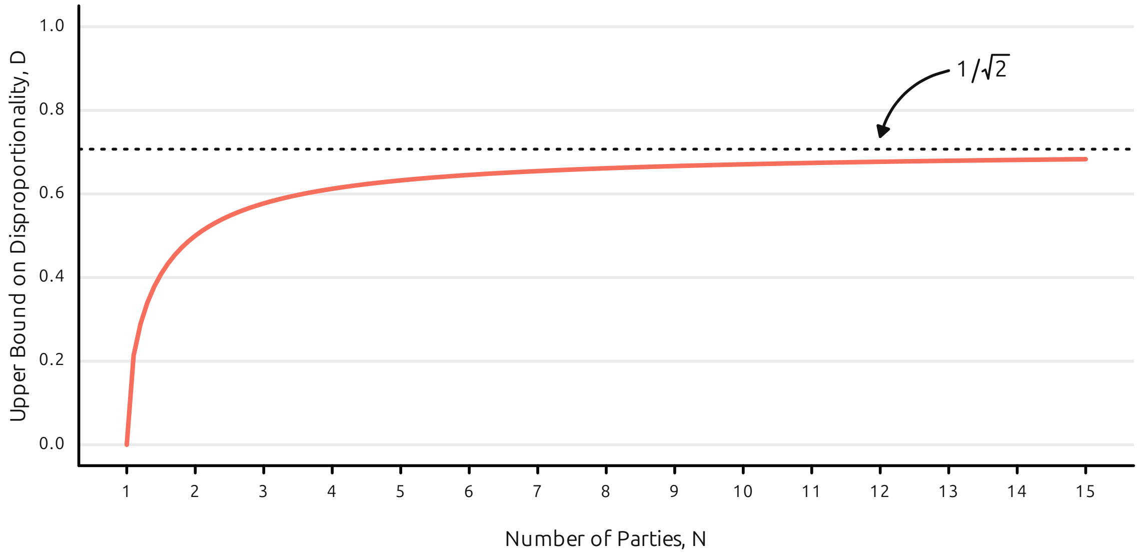

Figure 1: As the number of parties converges on ∞, the upper bound on the Gallagher index converges on $\frac{1}{\sqrt{2}}$. As such, where elections are free and fair and the electoral system majorizes votes, disproportionality cannot exceed this value.

Figure 1 shows how this upper bound changes as the number of parties increases. Where there is just one party, disproportionality is always 0. As the number of parties grows, the upper bound then explodes, then converges on around 0.7. In fact, the exact value that it reaches is $\frac{1}{\sqrt{2}}$, which occurs in the limiting case where $N \to \infty$. To see how, note that:

$$D_{max} = \sqrt{\frac{N-1}{2N}} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{N}} \tag{7}$$

So, as N tends to ∞, $\frac{1}{N}$ tends towards 0, giving in the limit where $N \to \infty$:

$$D_{max} = \frac{1}{\sqrt{2}}\sqrt{1-0} = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}} \tag{8}$$

Therefore proving that disproportionality cannot exceed $\frac{1}{\sqrt{2}}$ at democratic elections.[1]

3 A NORMALISED INDEX OF DISPROPORTIONALITY

Clearly, it is not desirable for the number of parties to cause variation in the Gallagher index even where electoral distortion is constant. For one, it makes it very difficult to make meaningful comparisons across time and space. After all, the number of parties determines the upper bound on the index, so apparent changes in disproportionality could, at least in part, be a function of party system fragmentation and not any electoral distortion. Likewise, the index's varying range also makes normative claims about what constitutes an "acceptable" level of disproportionality much less meaningful. For instance, one academic told Canada's

[1]Interestingly, Shugart and Taagepera (2017, 143) report a value of 0.707 as a coefficient on the *lower* bound of disproportionality when considering only the vote and seat shares of the largest party, such that $D_{min} = 0.707\,(s_1 - v_1)$. However, they do not link this coefficient to either the value $\frac{1}{\sqrt{2}}$ or to the more general case.

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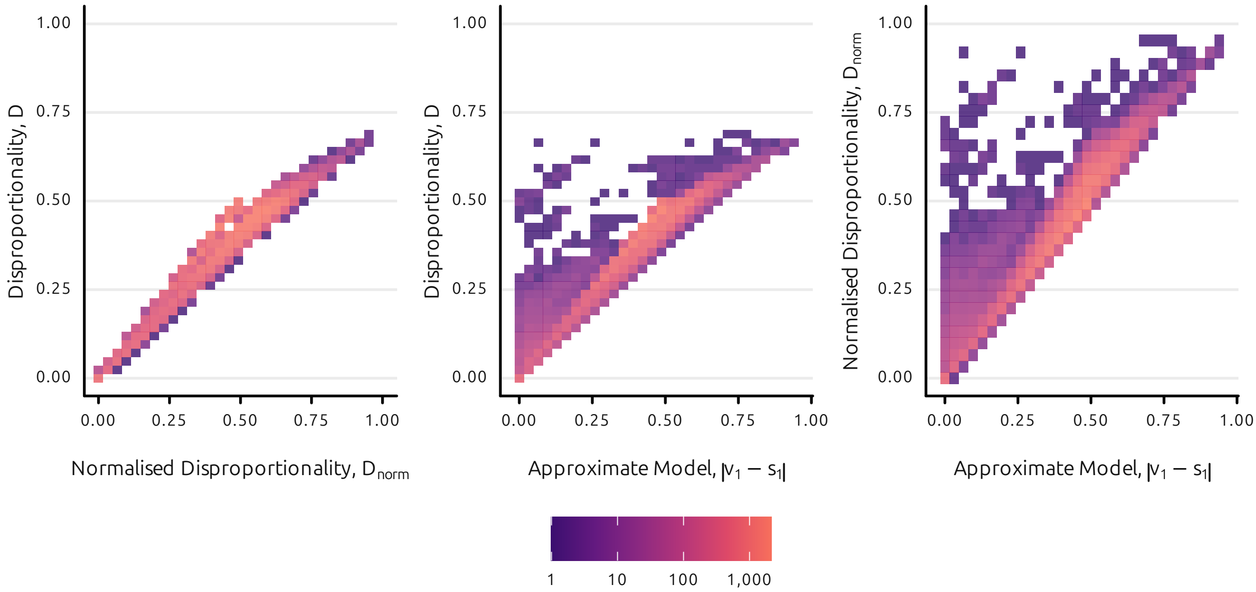

Figure 2: Disproportionality across 75,061 electoral districts at 543 elections in 95 countries between 1867 and 2022. The Gallagher index, D, shows wide variation at a given value of the normalised index, D_{norm}. Both indices track the largest party's vote–seat gap $|v_1 - s_1|$, though D_{norm} is bounded between 0 and 1, which aids in interpretation and prediction.

Special Committee on Electoral Reform that a Gallagher index of less than 0.05 was considered "excellent" (Special Committee on Electoral Reform 2016, 42). Yet this benchmark means very little if it implies a different level of disproportionality for different numbers of parties.

With this in mind, we might consider normalising the Gallagher index to output values from 0 (perfect *proportionality*) to 1 (maximal *disproportionality*) under democratic conditions no matter the number of parties. To do so, we can divide Equation 1 by Equation 6 to get:

$$D_{norm} = \sqrt{\frac{\frac{1}{2}\sum_{i=1}^{N}(v_i - s_i)^2}{\frac{N-1}{2N}}} \tag{9}$$

Simplifying the fraction inside this equation then gives:

$$\frac{\frac{1}{2}}{\frac{N-1}{2N}} = \frac{1}{2} \times \frac{2N}{N-1} = \frac{N}{N-1} \tag{10}$$

Which allows us to rewrite the normalised index of disproportionality as follows:

$$D_{norm} = \sqrt{\frac{N-1}{N}\sum_{i=1}^{N}(v_i - s_i)^2} \tag{11}$$

Figure 2 shows the distribution of disproportionality scores based on results from 75,061 electoral districts at 543 elections in 95 countries between 1867 and 2022 taken from the Constituency-Level Elections Archive (Kollman et al. 2024). In all cases, I use the district-level vote and seat shares to compute the Gallagher index, D, and my normalised disproportionality index, D_{norm}. As we can see from the left most panel, the Gallagher index, D, shows a great deal

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of variation at any given normalised value, D_{norm}. Indeed, where $D_{norm} = 0.5$, D varies by around 0.125. Further, this variation also appears to differ across the range of the normalised index, with lower values of D and D_{norm} tending to be much more similar.

Given Equations 1 and 11, we might expect the vote and seat shares that contribute the most to the index to belong to the first party. We might, therefore, predict D and D_{norm} using a simple model that relies on only the absolute difference between v_1 and s_1, such that $\hat{D} = \hat{D}_{norm} = |v_1 - s_1|$. The middle and right most panels of Figure 2 plot the Gallagher index, D, and my normalised disproportionality index, D_{norm}, against this quantity. Both show similar relationships, which makes sense since D_{norm} is simply a normalised version of D. Yet, compared to the relationship shown in the middle panel, the relationship between $|v_1 - s_1|$ and D_{norm} (right most panel) has an attractive property: since the index is normalised to range from 0 to 1, its lower bound is $|v_1 - s_1|$ rather than $\frac{1}{\sqrt{2}}|v_1 - s_1|$, making it simpler to predict.

4 CONCLUSION

Though political scientists often assume that the Gallagher index varies between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*), I show that this is not true for free and fair elections. Instead, I prove that the index is bounded above by $\sqrt{\frac{N-1}{2N}}$ and that its theoretical limit is $\frac{1}{\sqrt{2}}$, which occurs where the number of parties, N, tends to ∞. I then propose a new normalised measure, D_{norm}, that rescales Gallagher's index of disproportionality to fall between 0 and 1 under democratic conditions.

These contributions have implications for measurement, data management, and normative theory. On measurement, my normalised index removes distortions introduced into the Gallagher index by varying numbers of parties, allowing for more accurate comparisons across time and space. On data management, the upper bounds that I identify suggest that any values of the Gallagher index that exceed $\sqrt{\frac{N-1}{2N}}$ can be flagged as likely errors. Indeed, in writing this paper I used this approach to identify 19 errant cases in the Constituency-Level Election Archive data. Finally, on normative theory, researchers and practitioners may find my normalised measure useful when setting benchmarks for "acceptable" disproportionality, since it provides a scale-invariant reference. As such, the contributions that I make here not only clarifies the mathematical properties of a widely used index in political science but also offer tools to improve the quality of our democracies.

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