WORKING PAPER

Disproportionality at Fair Elections Cannot Exceed $\frac{1}{\sqrt{2}}$

A Note on the Gallagher Index and a New Normalised Measure

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ABSTRACT The Gallagher index of disproportionality is perhaps the most popular measure of electoral distortion in all of political science. While many assume that the index ranges from 0 (perfect proportionality) to 1 (perfect disproportionality), I prove that in any free and fair election its maximum possible value is bounded above by $\sqrt{\frac{N-1}{2N}}$, reaching a maximum of $\frac{1}{\sqrt{2}}$ where the number of parties, N, tends to infinity. This bound arises in the limiting case where all parties receive equal vote shares but only one secures representation. Building on this result, I then propose a new normalised version of the index that rescales disproportionality to lie between 0 and 1 under democratic conditions.

1 INTRODUCTION

The Gallagher index (1991) is now the most popular measure of disproportionality in political science. Recent studies have used it to investigate how disproportionality affects phenomena as varied as intra-party cohesion (Matakos et al. 2024), subjective health and well-being (Toshkov and Mazepus 2023), and corporate investment decisions (Amore and Corina 2021). Its impact has also transcended academic research. After then-Liberal Party leader Justin Trudeau committed to making 2015 Canada's "last federal election conducted under the first-past-the-post voting system" (Liberal Party of Canada 2016, 8), the Gallagher index informed the work of the Canadian House of Commons Special Committee on Electoral Reform (2016). Thus, the index's dual role, as empirical and practical tool, underlines its general importance.

In principle, the Gallagher index can vary between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*). However, in this paper, I prove that it is bound from above by $\sqrt{\frac{N-1}{2N}}$ at all free and fair elections. As the number of vote-winning parties, N, tends to infinity, this upper bound then converges on $\frac{1}{\sqrt{2}}$, setting a ceiling on the index under democratic conditions. This

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suggests that, unlike *ratio*-based measures of disproportionality, some variation in *difference*-based measures like the Gallagher index occurs due to the size of the party system and *not* electoral distortion. Since party system fragmentation has increased in recent years (Norris 2024), this poses a considerable problem if we want to compare disproportionality across time and space. To account for this, I introduce a new normalised version of the Gallagher index that varies between 0 and 1 at democratic elections, and no matter the number of parties.

2 MEASURING DISPROPORTIONALITY

Disproportionality—the degree to which parties' shares of seats diverge from their shares of votes—matters for both normative and empirical reasons. Normatively, disproportionality poses a challenge to the equality of representation: the idea that each vote cast carries a common weight. Indeed, McGann (2006) even goes so far as to argue that "proportionality is logically implied by the basic value of political equality, that is, by the concept of democracy itself" (35). Empirically, disproportionality arises as a consequence of electoral systems (Duverger 1964). Where district magnitudes are large and electoral thresholds are low, elections tend to produce more proportional outcomes (Shugart and Taagepera 2017; Anckar 1997).

In general, there are two distinct perspectives when it comes to measuring the disproportionality of electoral systems: one relative and one absolute. The *relative* perspective operationalises disproportionality in terms of the *seats-to-vote ratio*, the share of seats that some party won over its share of votes. The idea here is that—consistent with one-person-one-vote—disproportionality concerns the equal treatment of *voters* (Van Puyenbroeck 2008). If party A's voters get more seats per share of votes than voters for party B, then the latter are at a disadvantage. The *absolute* perspective, however, operationalises disproportionality in terms of the *absolute difference* between vote and seat shares. Consequently, it treats disproportionality as a phenomenon that affects *political parties*. Despite compelling arguments in favour of relative measures of disproportionality like the Sainte-Laguë index (see Goldenberg and Fisher 2019; Van Puyenbroeck 2008; Gallagher 1991), absolute measures of disproportionality are now much more common in the study of electoral systems.

The oldest absolute measure of disproportionality comes from Rae (1971) and equals the average absolute difference between the vote share, v, and seat share, s, of each party, N:

$$D_R = \frac{1}{N}\sum_{i=1}^{N}|v_i - s_i| \tag{1}$$

However, Rae's measure applies a downward bias for large party systems. To see why, consider that Rae's index tends to zero in the limit where the number of parties tends to infinity, implying perfect proportionality no matter the distribution of seats and votes among the major political parties (Katz 1980). To overcome this problem, Loosemore and Hanby (1971) modified Rae's index to ignore the number of parties, effectively treating disproportionality as an election-level phenomenon, and then multiplied the resulting index by $\frac{1}{2}$ to rescale its output from 0 (perfect *proportionality*) to 1 (perfect *disproportionality*):

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$$D_{LH} = \frac{1}{2}\sum_{i=1}^{N}|v_i - s_i| \quad (2)$$

Though the Loosemore-Hanby index represented a marked improvement on Rae's index and soon became the industry standard, it suffers from the opposite problem: it applies an *upward* bias for large party systems (Lijphart 1994). Likewise, it also fails Dalton's (1920) principle of transfers—the axiom that moving seats from overrepresented to under-represented parties should always cause disproportionality to fall—and treats electoral systems that use largest remainders as having more proportional outcomes by design (Gallagher 1991).

Gallagher's (1991) "least squares" index strikes a balance between these two extremes. Rather than rely on *absolute* difference between vote and seat shares—like Rae (1971) and Loosemore and Hanby (1971)—it relies on the *squared* differences between them instead, much like the method of least squares from which it takes its name. Furthermore, the index is also consistent with Dalton's (1920) principle of transfers and balances under- and over-sensitivity to the number of parties by registering "a few large discrepancies more strongly than a lot of small ones" (Gallagher 1991, 40). We compute the measure as follows:

$$D = \sqrt{\frac{1}{2}\sum_{i=1}^{N}(v_i - s_i)^2} \quad (3)$$

The Gallagher index now represents the "industry standard for electoral analysis" (Goldenberg and Fisher 2019, 203) and several seminal works in electoral studies have used it to evidence their claims. For instance, Lijphart (2012, 1994) investigates how different electoral systems in use worldwide produce different levels of disproportionality, amongst other factors, while Shugart and Taagepera (2017) provide an approximate predictive model of the index as part of their broader project on the institutional dynamics of electoral systems.

While the Gallagher index is undoubtedly popular, there is still a lot that we do not know about how the various measures of disproportionality work. A range of articles have attempted to address this gap in our knowledge (see Goldenberg and Fisher 2019; Van Puyenbroeck 2008; Taagepera and Grofman 2003). All tend to assume that the Gallagher index varies between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*). However, as I show in the next section, there is good reason to believe that this is not the case, at least when it comes to democratic elections.

3 PROOF THAT DISPROPORTIONALITY CANNOT EXCEED $\frac{1}{\sqrt{2}}$

In principle, the Gallagher index can take any value between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*). To be perfectly *proportional*, the distribution of vote and seat shares must be identical. For instance, two parties might receive 80% and 20% of all votes and 80% and 20% of all seats. Similarly, to be perfectly *disproportional*, one party must win *all* of the seats while other parties win *all* of the votes. Of course, such an outcome would be undemocratic: to ensure political equality, democratic electoral systems must not discriminate against voters or the choices they make (McGann 2006). Under democratic conditions, the most *disproportional* outcome instead occurs where all parties receive *exactly* the same share of the vote, but where only one party wins any seats. That is, where all $v_i = \frac{1}{N}$ while $s_1 = 1$ and all $s_{i \neq 1} = 0$. Here, the sum in the index simplifies to give a new upper bound.

First, rewrite the sum in Equation 3 to separate $i = 1$ from all other cases:

$$\sum_{i=1}^{N} (v_i - s_i)^2 = (v_1 - s_1)^2 + \sum_{i=2}^{N} (v_i - s_i)^2 \tag{4}$$

Then, substitute in $v_i = \frac{1}{N}$, $s_1 = 1$, and $s_{i \neq 1} = 0$, giving:

$$\left(\frac{1}{N} - 1\right)^2 + (N-1)\left(\frac{1}{N} - 0\right)^2 \tag{5}$$

Equation 5 then simplifies as follows:

$$\left(\frac{1}{N} - 1\right)^2 + (N-1)\left(\frac{1}{N} - 0\right)^2 = \frac{(N-1)^2}{N^2} + \frac{N-1}{N^2} = \frac{N-1}{N} \tag{6}$$

And, substituting Equation 6 for the sum in Equation 3, we get:

$$D_{max} = \sqrt{\frac{1}{2} \times \frac{N-1}{N}} \tag{7}$$

Which again simplifies to give the upper bound on D for N vote-winning parties:[1]

$$D_{max} = \sqrt{\frac{N-1}{2N}} \tag{8}$$

Figure 1 shows how the upper bound changes as the number of parties increases. As N grows, the upper bound grows too, before converging around 0.7. In fact, the exact value that it reaches is $\frac{1}{\sqrt{2}}$, which occurs in the limiting case where $N \to \infty$. To see how, note that:

$$D_{max} = \sqrt{\frac{N-1}{2N}} = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{1}{N}} \tag{9}$$

So, as N tends to ∞, $\frac{1}{N}$ tends towards 0, giving:

$$D_{max} = \frac{1}{\sqrt{2}}\sqrt{1-0} = \frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{2}} \tag{10}$$

Therefore proving that disproportionality cannot exceed $\frac{1}{\sqrt{2}}$ at democratic elections.[2]

[1] All indexes that operationalise disproportionality in terms of the difference between vote and seat shares face similar upper bounds. For the index that Rae (1971) proposes, $\frac{1}{N}\sum_{i=1}^{N}|v_i - s_i|$, it is $\frac{2(N-1)}{N^2}$. Likewise, for the index that Loosemore and Hanby (1971) propose, $\frac{1}{2}\sum_{i=1}^{N}|v_i - s_i|$, it is $\frac{N-1}{N}$. This is not true, however, for indexes that operationalise disproportionality in terms of the ratio of seats to votes, perhaps bolstering the argument that we should favour the Sainte-Laguë index instead (see Goldenberg and Fisher 2019; Van Puyenbroeck 2008).

[2] Interestingly, Shugart and Taagepera (2017, 143) report a value of 0.707 as a coefficient on the *lower* bound of disproportionality when considering only the vote and seat shares of the largest party, such that $D_{min} = 0.707\,(s_1 - v_1)$. However, they do not link this coefficient either to the value $\frac{1}{\sqrt{2}}$ or to the more general case.

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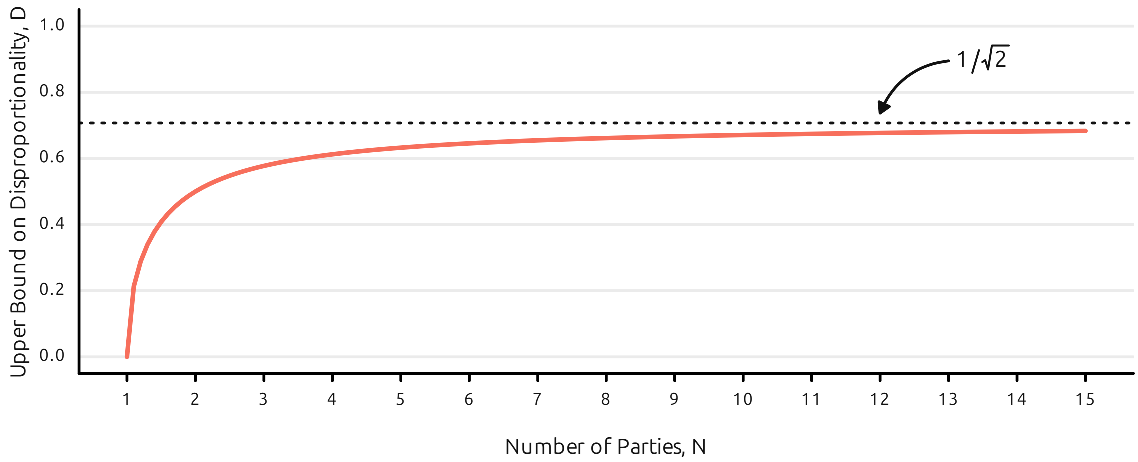

Figure 1: As the number of parties converges on ∞, the upper bound on the Gallagher index converges on $\frac{1}{\sqrt{2}}$. As such, where elections are free and fair and the electoral system majorizes votes, disproportionality cannot exceed this value.

4 A NORMALISED INDEX OF DISPROPORTIONALITY

As we have seen, the number of parties imposes a ceiling on the Gallagher index. Where N parties compete for election, the index varies between 0 (perfect *proportionality*) and $\sqrt{\frac{N-1}{2N}}$ (maximal *disproportionality*) up to a limit of $\frac{1}{\sqrt{2}}$. This presents some challenges. Empirically, it makes comparisons across time and space much more difficult. If the number of parties bounds the index from above, apparent changes in disproportionality could, at least in part, be a function of fragmentation and not electoral distortion. Normatively, it also makes claims about "acceptable" levels of disproportionality much less meaningful. For instance, one academic witness told Canada's Special Committee on Electoral Reform that a value of less than 0.05 was "excellent" (Special Committee on Electoral Reform 2016, 42). But if this benchmark has a different meaning for different numbers of parties, it probably means very little.

With this in mind, we might consider normalising the Gallagher index so that it outputs values from 0 (perfect *proportionality*) to 1 (maximal *disproportionality*), no matter the number of parties. To do so, we can divide Equation 3 by Equation 8, giving:

$$D_{norm} = \sqrt{\frac{\frac{1}{2}\sum_{i=1}^{N}(v_i - s_i)^2}{\frac{N-1}{2N}}} \tag{11}$$

Simplifying the fraction inside this equation then gives:

$$\frac{\frac{1}{2}}{\frac{N-1}{2N}} = \frac{1}{2} \times \frac{2N}{N-1} = \frac{N}{N-1} \tag{12}$$

Which allows us to write the normalised index as follows:

$$D_{norm} = \sqrt{\frac{N-1}{N}\sum_{i=1}^{N}(v_i - s_i)^2} \tag{13}$$

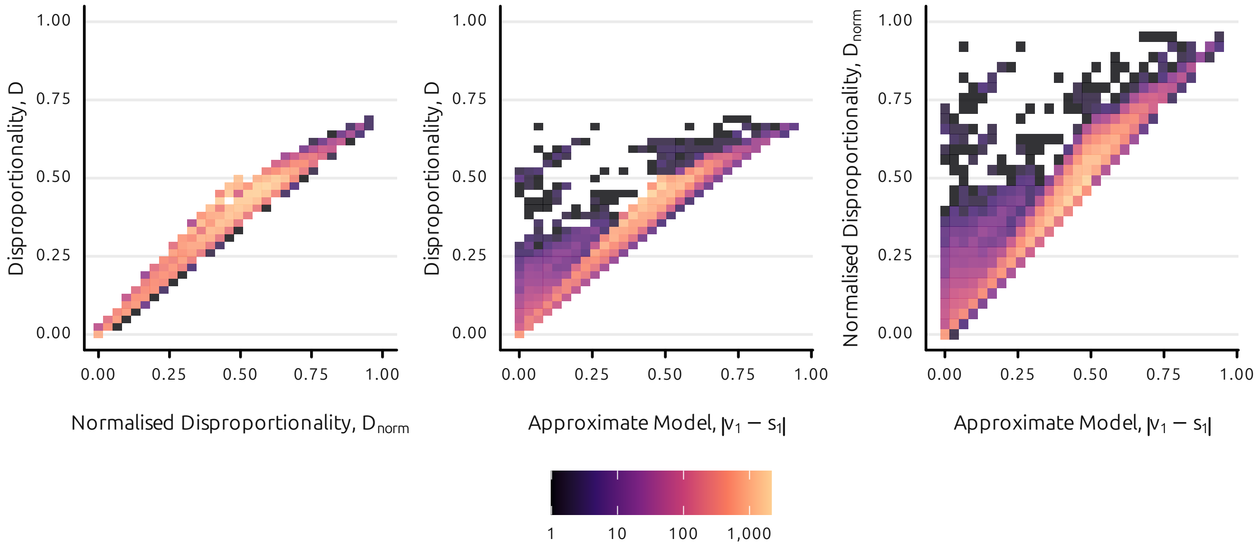

Figure 2: Disproportionality across 75,061 electoral districts at 543 elections in 95 countries from the Constituency-Level Elections Archive. The Gallagher index, D, shows wide variation at a given value of the normalised index, D_{norm}. Both indexes track the largest party's vote–seat gap $|v_1 - s_1|$, though D_{norm} is bounded between 0 and 1, which aids interpretation and prediction.

So that we might understand how my normalised index compares to the one that Gallagher (1991) proposes, Figure 2 shows the distribution of disproportionality scores based on results from 75,061 district-level competitions at 543 elections in 95 countries between 1867 and 2022 taken from the Constituency-Level Elections Archive (Kollman et al. 2024). In all cases, I use the district-level vote and seat shares to compute the Gallagher index, D, and my normalised index, D_{norm}. As the left hand panel makes clear, the Gallagher index, D, shows a great deal of variation at any given value on the normalised index, D_{norm}, evidencing the fact that the size of the party system contributes towards much of the variation in the Gallagher index. Indeed, where $D_{norm} = 0.5$, D varies by around 0.125—about one-eighth the range of the index. This variation also differs across the normalised index, with lower values tending to be more alike.

Given Equations 3 and 13, we might expect the vote and seat shares that contribute the most to the index to belong to the party that comes in first. Following Shugart and Taagepera (2017), we might, therefore, predict D and D_{norm} using a simple model that relies only on the absolute difference between v_1 and s_1, such that $\hat{D} = \hat{D}_{norm} = |v_1 - s_1|$. The middle and right hand panels of Figure 2 plot the Gallagher index, D, and my normalised disproportionality index, D_{norm}, against this simple model. Both show similar results. Yet, compared to the relationship shown in the middle panel, the relationship between $|v_1 - s_1|$ and D_{norm} (right hand panel) is much more attractive: since the index is normalised to range from 0 to 1, its lower bound is $|v_1 - s_1|$ rather than $\frac{1}{\sqrt{2}}|v_1 - s_1|$, making prediction more simple.

In order to determine the impact that the number of parties has when we make comparisons across time and space, Figure 3 plots the two indices for legislative elections in three countries—Australia, Ireland, and the Netherlands—that have experienced high, medium, and low levels of disproportionality since in the post-war period, respectively. In each case, my

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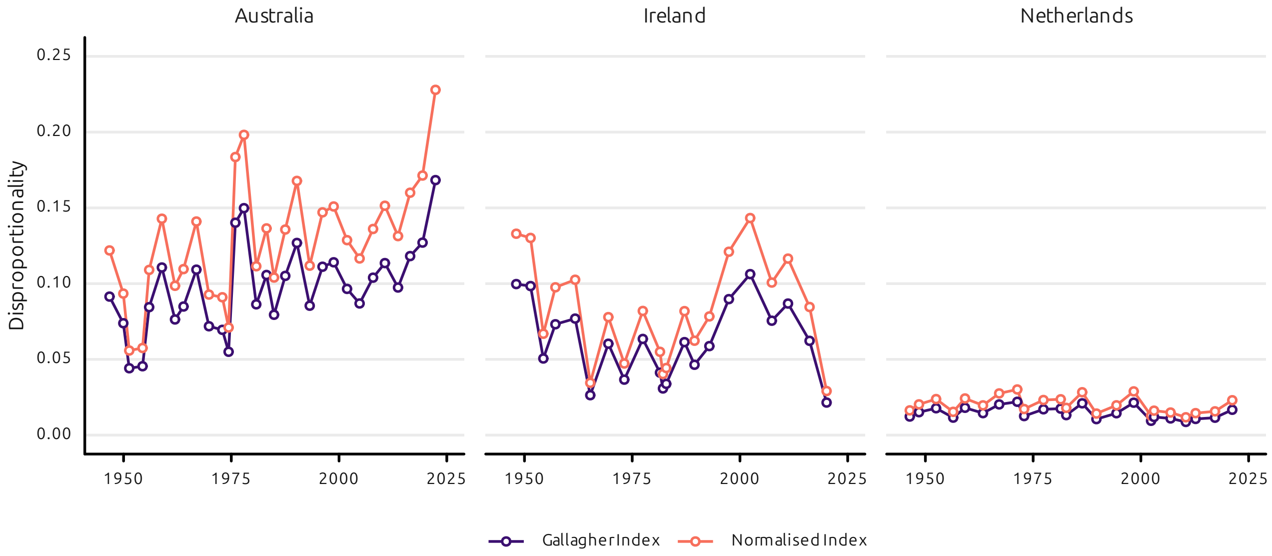

Figure 3: Disproportionality in three countries—Australia, Ireland, and the Netherlands—that exhibit high, medium, and low levels of disproportionality at legislative elections. Note that the difference between the Gallagher index, D, and my normalised index, D_{norm} gets larger as disproportionality grows. Data here come from ParlGov.

normalised disproportionality index, D_{norm}, produces values that are larger than the Gallagher index, D, much as we would expect. Yet, the difference between the two indexes becomes larger as disproportionality grows. Indeed, in the case of the Netherlands, the overall difference is negligible (MAE = 0.007). However, the difference is larger in Ireland (MAE = 0.02) and Australia (MAE = 0.03) and, in the latter case, the gap between the two measures grows up to 0.06 at more recent elections. To put this figure in context, the Gallagher index, D, has a mean of 0.08 and a standard deviation of 0.09 across all legislative elections in the ParlGov data set (Döring and Manow 2024). So this is no trivial difference: the largest value in the Australian case represents a 0.7 standard deviation change in the value of the Gallagher index.

5 CONCLUSION

Though political scientists often assume that the Gallagher index varies between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*), I show that this is not true under democratic conditions. Instead, I prove that the index is bounded from above by $\sqrt{\frac{N-1}{2N}}$ and that its theoretical limit is $\frac{1}{\sqrt{2}}$, which occurs where the number of parties, N, tends to ∞. I then propose a new normalised measure, D_{norm}, that rescales Gallagher's index of disproportionality to fall between 0 and 1 where elections are free and fair.

These contributions have implications for measurement, data management, and normative theory. On measurement, my normalised index removes distortions introduced into the Gallagher index by varying numbers of parties, allowing for more accurate comparisons across time and space. On data management, the upper bounds that I identify suggest that any values of the Gallagher index that exceed $\sqrt{\frac{N-1}{2N}}$ can be flagged as likely errors. Indeed, I used this approach to identify 19 errant cases in the Constituency-Level Election Archive data in the process of writing this paper. Likewise, when it comes to normative theory, researchers and practitioners may find my normalised measure useful when setting benchmarks for "acceptable" levels of disproportionality, since it provides a reference that does not vary with the size of the party system. As such, the contributions that I make here not only clarifies the mathematical properties of a widely used index in political science but also offer tools to improve the quality of our democracies.

More generally, however, my findings suggest that more work needs to be done both to conceptualise what disproportionality really *means* and the best way that it can be *measured*. Indeed, there is good reason to sympathise with Goldenberg and Fisher (2019), Van Puyenbroeck (2008), and others who argue persuasively in favour of alternative ratio-based measures like the one first proposed by Sainte-Laguë (1910). As well as satisfying Dalton's principle of transfers, the Sainte-Laguë index is not bound above by the number of parties.

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