WORKING PAPER

Disproportionality is Sensitive to the Number of Parties

A Note on the Gallagher Index and a New Normalised Measure

Jack Bailey

Department of Politics, University of Manchester, Oxford Road, Manchester, UK

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Correspondence: Blinded

ABSTRACT The Gallagher index is the most popular measure of disproportionality in political science. While many assume that it can range from 0 (perfect proportionality) to 1 (perfect disproportionality), I prove that its maximum value is constrained by the size of the party system. For any election which features N_v effective vote-winning parties, the Gallagher index cannot exceed $\sqrt{\frac{N_v+1}{2N_v}}$. In some cases, this is a problem, since it makes it difficult to compare disproportionality across time and space, especially given rising fragmentation. As such, I propose a new normalised index that runs from 0 to 1, no matter the size of the party system.

1 INTRODUCTION

The Gallagher index (1991) is the most popular measure of disproportionality in political science. Recent work has used it to investigate how disproportional elections affect phenomena as varied as intra-party cohesion (Matakos et al. 2024), subjective health and well-being (Toshkov and Mazepus 2023), and corporate investment decisions (Amore and Corina 2021). The index has had a life beyond academia. When Justin Trudeau, then leader of the Liberal Party, committed to making 2015 Canada's "last federal election conducted under the first-past-the-post voting system" (Liberal Party of Canada 2016, 8), the Gallagher index informed the work of the country's Special Committee on Electoral Reform (2016). This dual role—as empirical measure and practical tool—demonstrates the index's general importance.

In principle, the Gallagher index can vary between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*). However, in this paper, I show that its actual maximum is often much lower, since the size of the party system acts as a break on the range of values that it can take. To this end, I prove that, where there are N_v effective vote-winning parties, the Gallagher index has an upper bound of $\sqrt{\frac{N_v+1}{2N_v}}$. As the effective number of vote-winning parties tends

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to infinity, this bound then converges on $\frac{1}{\sqrt{2}}$. This is a problem for the comparative study of elections, since it suggests that our primary measure of disproportionality also responds to the number of parties. To account for this, I introduce a new normalised index that allows for comparisons of disproportionality across party systems of different sizes.

2 MEASURING DISPROPORTIONALITY

Disproportionality—the extent to which seat shares diverge from vote shares diverge—has both normative and empirical consequences. Normatively, disproportionality challenges the equality of representation: the notion that each vote should carry equal weight (Dahl 1971). Indeed, McGann even argues that "proportionality is logically implied by the basic value of political equality, that is, by the concept of democracy itself" (2006, 35). Empirically, disproportionality is a natural consequence of electoral system design, particularly the method of seat allocation (Duverger 1964). Where district magnitudes are large and electoral thresholds are low, elections tend to produce less disproportional outcomes than where they are not (Shugart and Taagepera 2017; Anckar 1997).

In general, there are two distinct perspectives on the measurement of disproportionality: one relative and one absolute. The *relative* perspective operationalises disproportionality in terms of the seats-to-vote ratio, the share of seats that some party wins divided by its share of the vote. The idea here is that—consistent with the equality of representation—disproportionality concerns the equal treatment of *voters* (Van Puyenbroeck 2008). If the same share of votes wins voters for party A more seats than voters for party B, then the latter are at a disadvantage. The *difference* perspective, instead, operationalises disproportionality in terms of the difference between vote and seat shares. Consequently, it treats disproportionality as a phenomenon that affects *political parties* instead. Despite compelling arguments in favour of relative measures of disproportionality like the Sainte-Laguë index (see Goldenberg and Fisher 2019; Van Puyenbroeck 2008; Gallagher 1991), absolute measures of disproportionality are now much more common in the study of electoral systems.

The oldest absolute measure of disproportionality comes from Rae (1971) and equals the average absolute difference between the vote share, v, and seat share, s, of all parties, N:

$$D_R = \frac{1}{N} \sum_{i=1}^{N} |v_i - s_i| \tag{1}$$

However, Rae's measure applies an implicit downward bias for large party systems. To see why, consider that Rae's index tends to zero in the limit where the number of parties tends to infinity, implying perfect proportionality no matter the distribution of seats and votes (Katz 1980). To overcome this problem, Loosemore and Hanby (1971) modified Rae's index to ignore the number of parties—effectively treating disproportionality as an election-level phenomenon—and then multiplied the resulting index by $\frac{1}{2}$ to rescale its output from 0 (perfect *proportionality*) to 1 (perfect *disproportionality*):

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$$D_{LH} = \frac{1}{2}\sum_{i=1}^{N}|v_i - s_i| \tag{2}$$

Though the Loosemore-Hanby index represented a marked improvement on Rae's design and soon became the industry standard, it suffers from the opposite problem: it applies an *upward* bias for large party systems instead (Lijphart 1994). It also fails Dalton's (1920) principle of transfers—the axiom that moving seats from over-represented to under-represented parties should always cause disproportionality to fall—and treats largest remainders systems as having more proportional outcomes by design (Gallagher 1991).

Gallagher's (1991) 'least squares' index attempts to strike a balance between these extremes. Rather than rely on the *absolute* difference between vote and seat shares—like Rae (1971) and Loosemore and Hanby (1971)—it relies on the *squared* difference instead, much like the method of least squares from which it takes its name. Gallagher's index is also consistent with Dalton's (1920) principle of transfers and balances under- and over-sensitivity to the number of parties by registering "a few large discrepancies more strongly than a lot of small ones" (Gallagher 1991, 40). We can compute the index as follows:

$$D = \sqrt{\frac{1}{2}\sum_{i=1}^{N}(v_i - s_i)^2} \tag{3}$$

The Gallagher index now represents the "industry standard for electoral analysis" (Goldenberg and Fisher 2019, 203), and several seminal works in electoral studies have used it to evidence their claims. For instance, Lijphart (2012, 1994) investigates how different electoral systems in use worldwide produce different levels of disproportionality. Likewise, Shugart and Taagepera (2017) provide an approximate predictive model of the index as part of their broader project on the institutional dynamics of electoral systems.

Though the Gallagher index is now the "industry standard", there is still much that we do not know about how it actually works. Consequently, a range of articles have attempted to address this gap in our knowledge (most notably Goldenberg and Fisher 2019; Van Puyenbroeck 2008; Taagepera and Grofman 2003). However, much like the rest of the literature, this work assumes that the Gallagher index can range from 0 (perfect *proportionality*) and 1 (perfect *disproportionality*). While this is true *in principle*, it is almost certainly not true *in practice*. This is because, as I show in the next section, the Gallagher index is sensitive to the overall size of the party system. And, as these systems grow, they enforce strict constraints on the values that the index may take.

3 ESTABLISHING AN UPPER BOUND ON THE GALLAGHER INDEX

To show that the size of the party system constrains the Gallagher index, first recall that we can characterise *any* party system, no matter how large it may be, in terms of the *effective number of parties* (Laakso and Taagepera 1979). This quantity, most often used in studies of party

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system fragmentation (e.g. Valentim and Dinas 2024), measures "the number of hypothetical *equal*-size parties that would have the same total *effect* on fractionalization of the system as have the actual parties of *unequal* size" (Laakso and Taagepera 1979, 4, emphasis in original). Where there are N *actual* vote-winning parties, we can compute the *effective* number of vote-winning parties, N_v, over some distribution of vote shares, v, such that:

$$N_v = \frac{1}{\sum_{i=1}^{N} v^2} \tag{4}$$

Next, recall that the Gallagher index can range from 0 (perfect *proportionality*) to 1 (perfect *disproportionality*). To be perfectly *proportional*, the distribution of vote and seat shares must be the same. This is always possible, no matter the size of the party system, so long as the number of seats is sufficiently large. To be perfectly *disproportional*, however, one party must win all of the seats while another party wins all of the votes. As such, this outcome is possible only in one specific circumstance: where $v_1 = 1$ and all other $v_i = 0$. Thus, by Equation 4, this implies that we can only maximise disproportionality where $N_v = 1$. Where party systems are any larger, the upper bound must take some lesser value.

Now consider that to maximise disproportionality for any fixed distribution of vote shares, v, we must allocate all of the seats to whichever party receives the least votes. For the sake of exposition, we will assume that this occurs where $i = 1$. Next, note that we can rewrite the sum in Equation 3 to separate out party $i = 1$ from all other parties:

$$\sum_{i=1}^{N} (v_i - s_i)^2 = (v_1 - s_1)^2 + \sum_{i=2}^{N} (v_i - s_i)^2 \tag{5}$$

Further, we know that party $i = 1$ must win all of the seats, such that $s_1 = 1$. We also know, therefore, that all other parties must not win any seats at all, such that $s_{i\neq 1} = 0$. This gives:

$$(v_1 - 1)^2 + \sum_{i=2}^{N} (v_i - 0)^2 \tag{6}$$

After expanding out the brackets and simplifying, the equation reduces to:

$$1 - 2v_1 + \sum_{i=1}^{N} v_i^2 \tag{7}$$

And, by the definition of the effective number of vote-winning parties, shown in Equation 4, we know that $\sum_{i=1}^{N} v_i^2 = \frac{1}{N_v}$, which we can substitute to get:

$$1 - 2v_1 + \frac{1}{N_v} \tag{8}$$

By substituting this for the sum in Equation 3, we arrive at an upper bound on the Gallagher index for any party system of a fixed size, N_v, with a smallest vote share, v_1:

$$D_{max} = \sqrt{\frac{1}{2}\left(1 - 2v_1 + \frac{1}{N_v}\right)} \tag{9}$$

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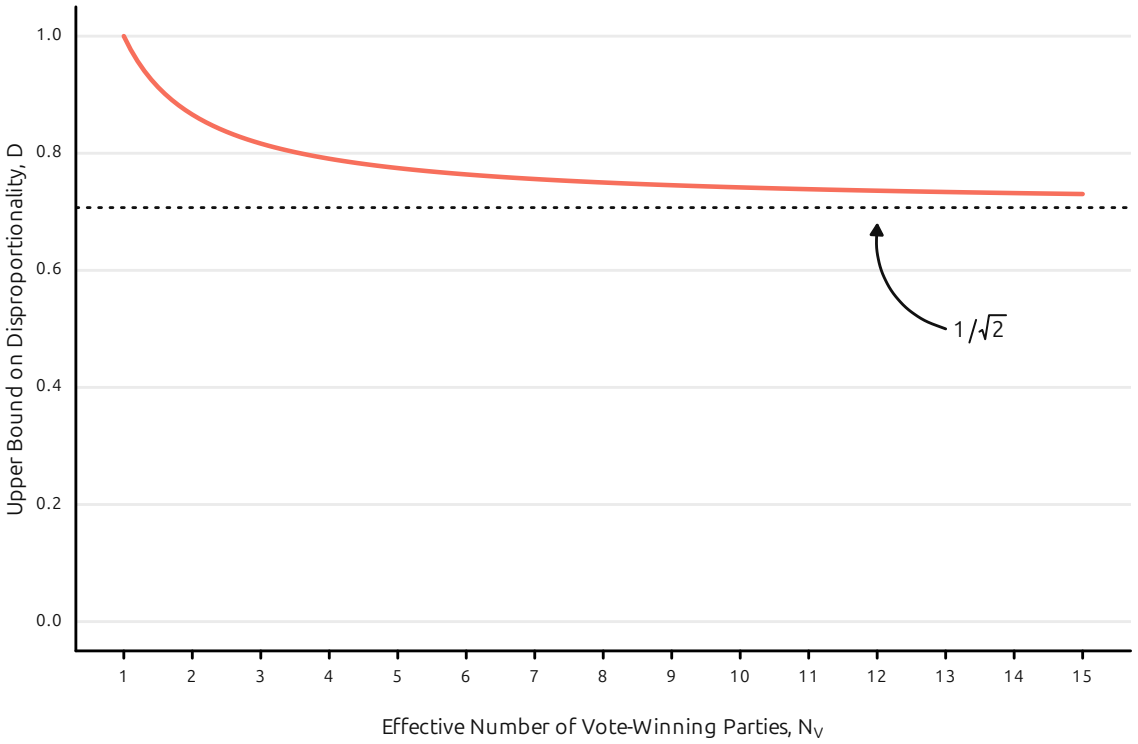


Figure 1: Where the effective number of vote-winning parties tends to ∞, the upper bound on the Gallagher index tends to $\frac{1}{\sqrt{2}}$. As such, the index is sensitive to the size of the party system.

This then implies that, in the most disproportional case, where $v_1 = 0$, the upper bound on the Gallagher index for any party system with N_v effective vote-winning parties must be:

$$D_{max} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{N_v}\right)} = \sqrt{\frac{N_v + 1}{2N_v}} \tag{10}$$

Figure 1 shows how this upper bound changes as the effective number of vote-winning parties increases. As N_v grows, the ceiling on the Gallagher index shrinks, condensing the possible range of disproportionality scores. The bound then converges on $\frac{1}{\sqrt{2}}$, which occurs in the limiting case where $N_v \to \infty$. To see how, recall from Equation 10 that:

$$D_{max} = \sqrt{\frac{1}{2}\left(1 + \frac{1}{N_v}\right)} \tag{11}$$

So, as N_v tends to ∞, $\frac{1}{N_v}$ tends towards 0, giving:[1]

$$D_{max|N_v \to \infty} = \sqrt{\frac{1}{2}(1 + 0)} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \tag{12}$$

[1]Interestingly, Shugart and Taagepera (2017, 143) report a value of 0.707 as a coefficient on the *lower* bound of disproportionality when considering only the vote and seat shares of the largest party, such that $D_{min} = 0.707\,(s_1 - v_1)$. However, they do not link this coefficient either to the value $\frac{1}{\sqrt{2}}$ or to the more general case.

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4 HOLDING THE INFLUENCE OF THE PARTY SYSTEM'S SIZE CONSTANT

In many cases, any upper bound on the Gallagher index will present both empirical and normative challenges. Empirically, it makes comparing elections more difficult. Since the size of the party system constrains the Gallagher index, apparent changes in disproportionality could arise due to changes in party system fragmentation instead. Normatively, it makes it much harder to argue that there is some 'acceptable' level of disproportionality. For instance, one expert witness told Canada's Special Committee on Electoral Reform that a value less than 0.05 was "excellent" (2016, 42). But if this benchmark has a different meaning for different sized party systems, then it probably means very little.

If we assume a difference-based approach to measuring disproportionality, then we must accept that disproportionality is, at least in part, a function of party system fragmentation. Quite apart from the bound that I identify above, this is necessary since maximising these indexes requires making the difference between some party's vote and seat shares, $v_i - s_i$, as large as possible. But, by *increasing* the size of v_i, we necessarily *decrease* the effective number of vote-winning parties and vice-versa. The two are, therefore, related.

Even so, it may sometimes be useful to hold the impact of the party system's size constant to compare the impact of seat allocation mechanisms alone. To this end, just as Gallagher (1991) adapted the Loosemore-Hanby index, we might adapt the Gallagher index to ensure that it always outputs values from 0 (perfect *proportionality*) to 1 (maximal *disproportionality*), no matter how many effective vote-winning parties there are. To do this, we need only divide Equation 3 by Equation 10, giving:

$$D_{norm} = \sqrt{\frac{\frac{1}{2}\sum_{i=1}^{N}(v_i - s_i)^2}{\frac{N_V+1}{2N_V}}} \tag{13}$$

Simplifying the fraction inside this equation then gives:

$$\frac{\frac{1}{2}}{\frac{N_V+1}{2N_V}} = \frac{1}{2} \times \frac{2N_V}{N_V+1} = \frac{N_V}{N_V+1} \tag{14}$$

Which allows us to write the normalised index as follows:

$$D_{norm} = \sqrt{\frac{N_V}{N_V+1}\sum_{i=1}^{N}(v_i - s_i)^2} \tag{15}$$

Figure 2 shows the distribution of disproportionality scores from 51,490 district-level contests in 'simple electoral systems'—single tier elections using first-past-the-post or party list proportional representation (Shugart and Taagepera 2017)—at 658 elections in 114 countries between 1900 and 2024, taken from The Elections Archive (Kollman et al. 2025). As the left hand panel makes clear, the Gallagher index, D, shows a good degree of variation for any given value on the normalised index, D_{norm}, evidencing the impact of the party system. Indeed, where $D_{norm} = 0.5$, D varies by around 0.125—one-eighth the possible range.

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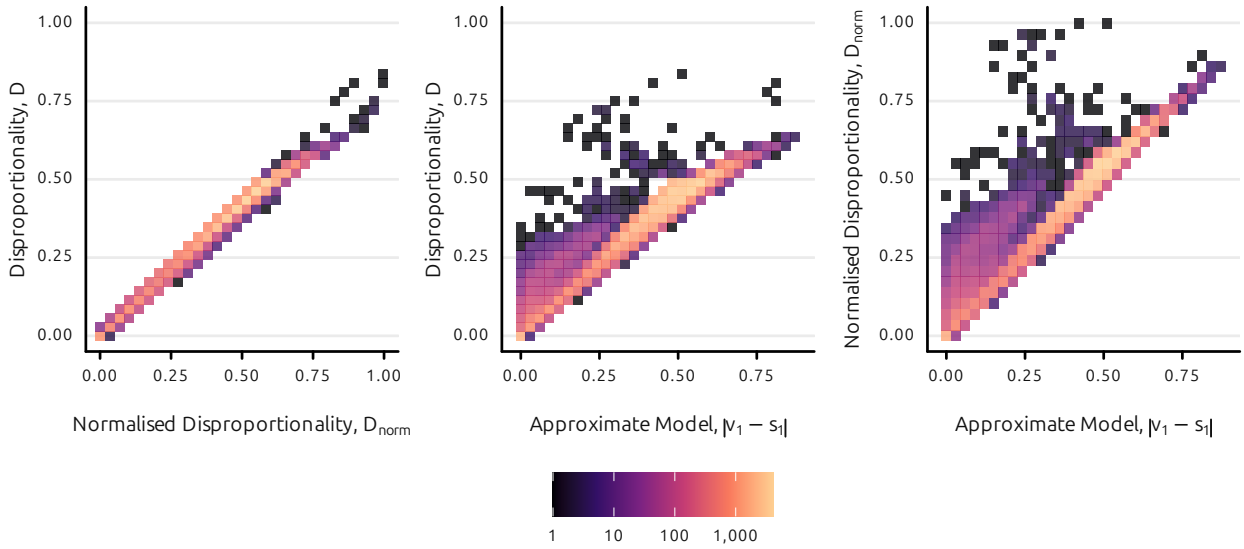


Figure 2: Disproportionality across 51,490 electoral districts at 658 elections in 114 countries from the Constituency-Level Elections Archive. The Gallagher index, D, shows wide variation at a given value of the normalised index, D_{norm}. Both indexes track the largest party's vote–seat gap $|v_1 - s_1|$, though D_{norm} does so spanning the full range between 0 and 1.

Given Equations 3 and 15, and for the reasons that I set out above, we might expect the first-placed party to have the greatest impact on the indexes. Thus, following Shugart and Taagepera (2017), I predict both D and D_{norm} using a simple model where $\hat{D} = \hat{D}_{norm} = |v_1 - s_1|$. The middle and right hand panels of Figure 2 plot D and D_{norm} against the model's predictions and both show similar results. However, holding constant the impact of the party system also introduces an attractive feature for D_{norm} (right hand panel): its lower bound is $|v_1 - s_1|$ rather than $\frac{1}{\sqrt{2}}|v_1 - s_1|$, making prediction somewhat simpler.

To determine how holding the size of the party system constant affects comparisons across time and space, Figure 3 uses data from ParlGov (Döring and Manow 2024) to plot the two indexes across legislative elections in Australia, Ireland, and the Netherlands—three countries that have experienced high, medium, and low disproportionality since the end of the Second World War. Much as we would expect given Equation 10, the normalised disproportionality index, D_{norm}, produces values that are larger than the Gallagher index, D. The difference between the two also grows as each increases. Since results in the Netherlands are extremely proportional, the overall difference in this case is negligible (MAE = 0.01). However, in Ireland (MAE = 0.03) and Australia (MAE = 0.06) the differences are larger. Indeed, in the latter case, the gap between the two measures grows to 0.1 at the most recent election. To put this figure in perspective, the Gallagher index, D, has a standard deviation of 0.09 across all legislative elections in the ParlGov data. Clearly, then, the impact of party system size is not trivial: in the Australian case, the difference between D and D_{norm} represents a 1 standard deviation change in the value of the Gallagher index.

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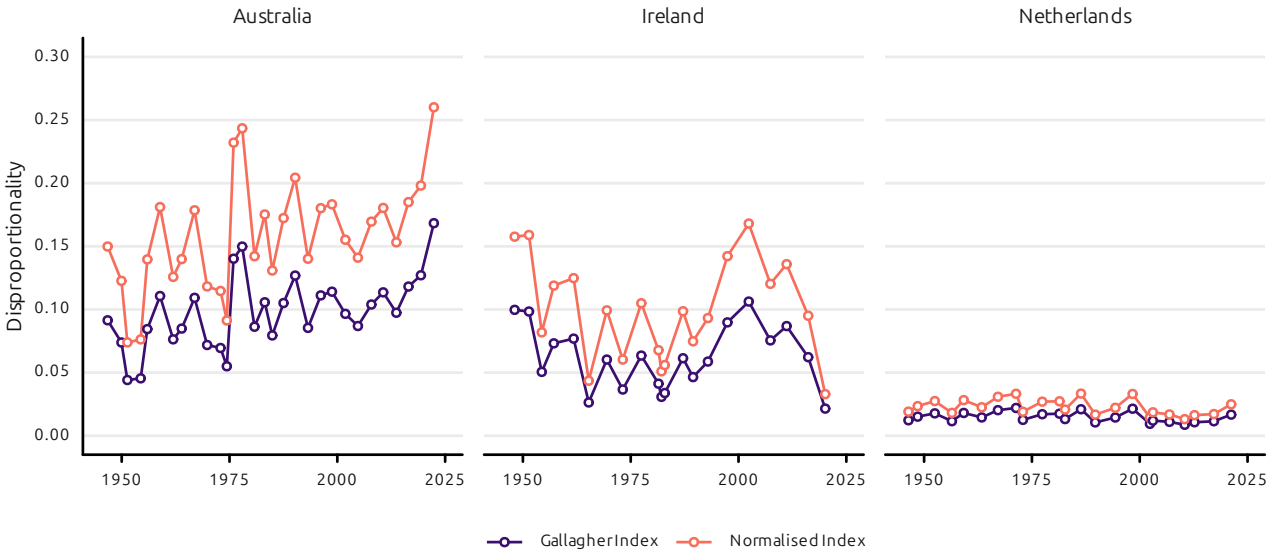


Figure 3: Disproportionality in three countries—Australia, Ireland, and the Netherlands—that exhibit high, medium, and low levels of disproportionality at legislative elections. Note that the difference between the Gallagher index, D, and the normalised index, D_{norm} grows as both indexes increase. Data here come from ParlGov.

5 CONCLUSION

Though political scientists often assume that the Gallagher index varies between 0 (perfect *proportionality*) and 1 (perfect *disproportionality*), I show that this is almost never the case. Instead, I prove that the size of the party system as measured in effective vote-winning parties, N_V, constrains the index from above such that it can reach a maximum of $\sqrt{\frac{N_V+1}{2N_V}}$. I then propose a new normalised index that falls between 0 and 1 no matter the size of the party system.

My findings have implications for measurement, data management, and normative theory. On measurement, the normalised index that I propose removes variation in the Gallagher index caused by party system fragmentation, allowing one to compare disproportionality across time and space. On data management, the upper bound that I identify suggests that any values of the Gallagher index that exceed $\sqrt{\frac{N_V+1}{2N_V}}$ can be flagged as probable errors. On normative theory, researchers and practitioners may find the normalised index useful when setting benchmarks for "acceptable" levels of disproportionality, since it provides a reference that does not vary according to the size of the party system. As such, the contributions that I make not only clarify the mathematical properties of a popular index in political science but also offer tools to improve the quality of our democracies.

More generally, however, my findings suggest that more work needs to be done both to conceptualise what disproportionality really means and how it should best be measured. There is good reason to sympathise with Goldenberg and Fisher (2019), Van Puyenbroeck (2008), and others who argue persuasively in favour of alternative relative measures like the one first proposed by Sainte-Laguë (1910). As well as satisfying Dalton's principle of transfers, the Sainte-Laguë index is unaffected by the size of the party system.

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