

WORKING PAPER

Estimating Effective Electoral Thresholds

A Logical Model

Jack Bailey

Department of Politics, University of Manchester, Oxford Road, Manchester, UK

This version: August 31, 2026

Correspondence: jack.bailey@manchester.ac.uk

ABSTRACT Electoral thresholds determine which parties win seats. Yet existing models of the effective threshold—the vote share at which a party has an equal chance of winning its first seat in a district—rest on strong simplifying assumptions. I develop a new logical model that derives the effective threshold from two electoral quantities: the district magnitude and the effective number of vote-winning parties. A non-linear Bayesian model fit to historic electoral data corroborates my theoretical logic and outperforms three existing alternatives in both explanatory power and out-of-sample prediction. The results show that effective thresholds decline with both district magnitude and electoral fragmentation. The model extends the Taageperan programme of parsimonious, interlocking electoral models and provides theoretically grounded propensity scores for studying the causal effects of representation.

1 INTRODUCTION

Electoral systems differ in how they treat smaller parties (Duverger 1964). Majoritarian systems are more punitive, while proportional systems are more permissive, since the two have different *electoral thresholds*: the vote share required for a party to win its first seat in a district. In some cases, these thresholds are written in law. Spain, for example, mandates that any party must gain at least 3% of all votes in a district before it may win any seats. In other cases, thresholds are a natural consequence of the system's mechanics instead (Rae, Hanby, and Loosemore 1971; Rokkan 1968). Whatever the case, the conclusion remains much the same: electoral thresholds determine which parties gain representation.

In this research note, I present a new logical model of the *effective electoral threshold*: the vote share at which any party has an equal chance of winning its first seat in a district. My model is both parsimonious and predictive: it uses just three variables and a single

parameter, while also making much more accurate out-of-sample predictions than the next two leading alternatives (Taagepera 1998; Lijphart 1994). My findings contribute to the Taagepera research agenda of interlocking electoral models (Taagepera 2008), but also have broader implications for electoral administration and electoral studies. First, my model should permit democratic designers to fine-tune effective thresholds to control against rising fragmentation without needing to enshrine arbitrary thresholds in law. Second, my model's predictions should also provide theoretically-grounded propensity scores that other political scientists can use to make causal inferences about what happens when parties win seats.

2 REPRESENTATION AND EFFECTIVE ELECTORAL THRESHOLDS

Even in countries that lack legal thresholds, parties face two electoral cut-offs. One is the *threshold of representation*, t_R : the *smallest* vote share that a party could gain while still having a chance to win a seat in a district. Consider a single-member district where four parties compete for election. Here, the threshold of representation would occur where all parties receive equal support, save for one that wins an additional vote (i.e. $t_R = 25\% + 1$). The other is the *threshold of exclusion*, t_X : the *largest* vote share that a party could gain while still having the chance *not* to win a seat in a district. Using the example above, the threshold of exclusion would occur where one party won one vote less than 50%, since any other party could still—at least in principle—beat it by winning 50% plus one vote (i.e. $t_X = 50\% - 1$).

Since the late 1960s, a broad literature has arisen regarding electoral thresholds. Building on preliminary research by Rokkan (1968), Rae, Hanby, and Loosemore (1971) defined thresholds of representation and exclusion for four common electoral systems, Grofman (1975, 311–15) added three more, while Lijphart and Gibberd (1977) corrected errors in the original formulas. More recent work has built on the foundation that these earlier scholars provided. For example, Ruiz-Rufino (2011, 2007) has shown that we can aggregate thresholds of representation and exclusion across electoral districts to characterise the performance of any electoral system. Likewise, legal thresholds, thresholds of representation, and thresholds of exclusion now provide a source of plausibly-exogenous variation in many regression discontinuity designs (e.g. Dinas, Riera, and Roussias 2015).

One part of this literature focuses on the *effective electoral threshold*: the point between t_R and t_X where any given party has an equal chance of winning its first seat in a district (Taagepera 1998; Lijphart 1994; Taagepera and Shugart 1989). This quantity is important for two reasons. First, unlike thresholds of representation or exclusion, which describe extreme conditions where winning a seat becomes possible or is guaranteed, the effective threshold provides a single point at which representation becomes more likely than not. In practice, this is often easier to work with than either alternative threshold while also corresponding more closely to the intuitive idea of a general electoral cut-off. Second, we can also use effective thresholds to characterise how permissive seat allocation methods are within any electoral district. As such, effective thresholds also provide political scientists with a common basis for comparing the performance of different electoral systems.

There are two leading models of the effective threshold: one by Lijphart (1994) and one by Taagepera (1998). Both have a similar provenance and are limited to simple electoral systems.¹ To derive both models, we start with the smallest threshold of representation, t_r , that occurs in these systems. Taagepera and Shugart (1989) show that, for any district with m seats and n_{v0} vote-winning parties, this corresponds to the lower threshold of the Hare formula:

$$t_r \geq \frac{1}{mn_{v0}} \quad (1)$$

Next, we pick the largest threshold of exclusion, t_x . Again, Taagepera and Shugart (1989) show that this threshold is common to all simple electoral systems and must be less than or equal to the Droop quota:

$$t_x \leq \frac{1}{m+1} \quad (2)$$

The thresholds of representation and exclusion represent the lower and upper bounds on the effective threshold, t , respectively. As such, we know that $t_r \leq t \leq t_x$. However, we do not know where t lies between them. Thus, Lijphart (1994) suggests taking the average to get the two thresholds' midpoint:

$$t = \frac{0.5}{m+1} + \frac{0.5}{mn_{v0}} \quad (3)$$

However, he also argues both that the lower threshold is too low and should instead be one-half the Hare quota and that we can use the district magnitude to approximate the number of parties, giving a new lower bound of $t_r = 1/(2m)$. Taking the average of this new lower bound and the upper bound in Equation 2 gives Lijphart's (1994) model of the effective threshold:

$$t = \frac{0.5}{m+1} + \frac{0.5}{2m} \quad (4)$$

Taagepera (1998) simplifies this equation, noting that we can rewrite it as:

$$t = \frac{0.25\left(3 + \frac{1}{m}\right)}{m+1} \quad (5)$$

Which, if we omit $1/m$, since it contributes only at small values of m , gives his model:

$$t = \frac{0.75}{m+1} \quad (6)$$

However, these approaches are not without criticism (see, for example, Bischoff 2009). One problem stems from the simplifying assumption that the number of vote-winning parties, n_{v0} , equals the district magnitude, m . To see why, consider a simple contest between two parties in a single-member district (i.e. $n_{v0} = 2, m = 1$). Here, both the threshold of representation and the threshold of exclusion equal 50%. The effective threshold *must*, therefore, be 50%. While

¹'Simple' electoral systems allocate seats within districts, use categorical ballots, and take place in a single round (see Shugart and Taagepera 2017). For example, first-past-the-post and districted systems of proportional representation are simple, but mixed-member systems or ranked-choice voting are not.

Lijphart's model gets the answer right, Taagepera's suggests that it is 37.5%. Furthermore, the assumption that both share—that we should weight the thresholds of representation, t_r , and exclusion, t_x , equally to compute the effective threshold, t —still remains to be tested and, as I will argue below, there is reason to believe that we can do better.

3 MODELLING EFFECTIVE THRESHOLDS

The thresholds of representation, t_r , and exclusion, t_x , are non-linear functions of the district magnitude, m , and the number of vote-winning parties, n_{v0} . Thus, any model of the effective threshold, t , will almost certainly be non-linear too. With this in mind, I follow a two-stage approach inspired by Shugart and Taagepera (2017), who advocate in favour of theoretically informed, parsimonious, and interlocking non-linear models of electoral systems. In the first stage, I use simple combinatorics to derive a model that predicts the effective threshold in the absence of all other data. In the second stage, I specify a corresponding Bayesian model which I fit using the *brms* package (Bürkner 2017) in R to test my logic using real world data.

3.1 THEORETICAL MODEL

Imagine an election in some electoral district. However, assume that you know *nothing* about the area that it covers, the people who live there, their beliefs, or their customs. Instead, assume that you know *only* that it has some magnitude, m , and will produce some *effective* number of vote-winning parties, n_{v2} . Given this information alone, what is the probability that any party will win at least one seat, s , in the district?

While this may seem an impossible task, simple combinatorics offers a potential solution. By design, the effective number of parties measures how many hypothetical and equally-large parties would produce a given level of fragmentation (Laakso and Taagepera 1979, 4). Since these hypothetical parties are equal in size, it follows that each would have a $1/n_{v2}$ chance of winning a given seat. Thus, if there is a $1/n_{v2}$ chance that a party *will* win a seat, there must be a $1 - 1/n_{v2}$ chance that it *will not*. The chance that a party will fail to win *any* of the district's m seats must therefore be $(1 - 1/n_{v2})^m$ and, so, the chance that this *will not* happen, and that any given party will win *at least one* seat in the district, which I denote θ , must be:

$$\Pr(s_i \geq 1 \mid m, n_{v2}) = \theta \approx 1 - \left(1 - \frac{1}{n_{v2}}\right)^m \quad (7)$$

Of course, simple combinatorics might work when it comes to rolling a die or randomly throwing balls into urns, but allocating seats to parties is different. Seat allocation methods are complex, often take place in several rounds, and a party's chance of winning a seat depends on all other parties' results. Nevertheless, I argue that this model still provides a reasonable approximation to the truth in the absence of all other information. First, the equal distribution over n_{v2} preserves both the overall degree of fragmentation and the probability that any two randomly selected votes will support the same party. Second, since effective parties are

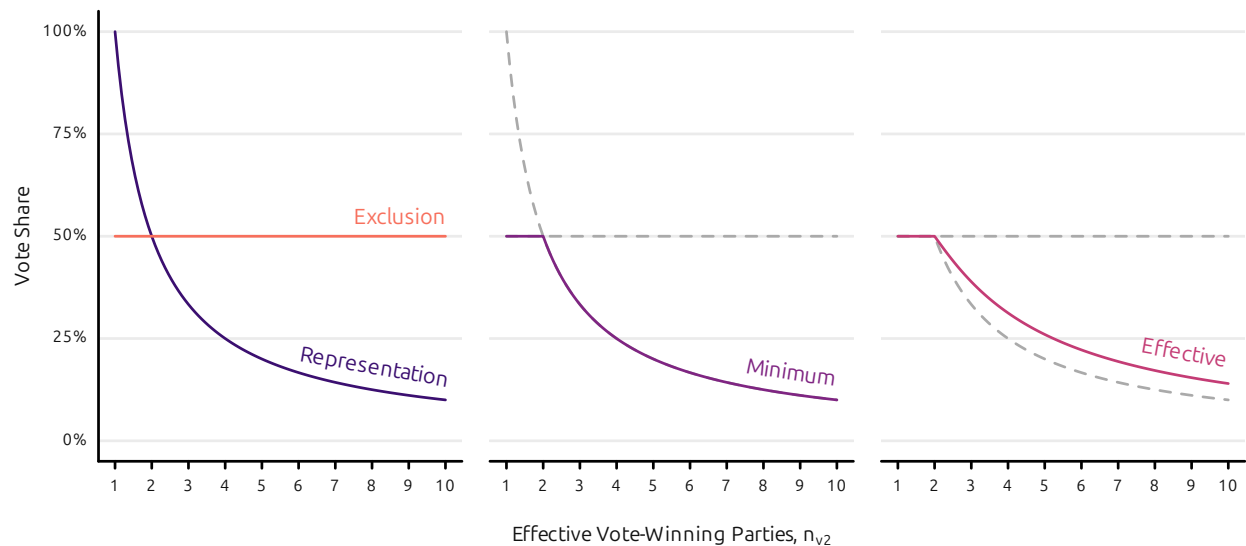


Figure 1: The thresholds of representation and exclusion (left panel) bound the effective threshold. However, since the threshold of representation cannot exceed the threshold of exclusion, their minimum is the true lower bound (middle panel). Weighting these bounds according to the probability that any party will win at least one seat, we get the effective threshold (right panel). Note that, in this case, the district magnitude is held constant at one.

exchangeable and equal in size, no dependences exist between them, so the least committal choice is to treat each as being independent. Finally, and perhaps most importantly, work by Hanretty (2022) uses a similar logic and shows that, while his model may be somewhat stylised, it makes sensible predictions nonetheless.

We can use this preliminary model to estimate the effective threshold, t . First, consider its bounds. The threshold of exclusion, t_x , depends only on the district magnitude, m , so is not sensitive to the number of parties. The threshold of representation, t_r , however, is a function of *both* the district magnitude, m , and the number of vote-winning parties, n_{v0} . Recall that t_r occurs where all vote-winning parties have an *equal* share of the vote, save for one that receives one additional vote. Typically, we ignore this single vote on the assumption that it contributes next to nothing to the overall share. Thus, since all parties receive the same share of the vote in this case, the *actual* and *effective* number of vote-winning parties coincide, suggesting that we can substitute effective for absolute parties, such that $t_r = 1/(mn_{v2})$.

Where a single party competes, thresholds remain undefined. Consequently, if we compute the threshold of representation, t_r , using *actual* numbers of parties, its largest value occurs where $n_{v0} = 2$ and $m = 1$, giving $t_r = 1/2$. However, this is *not* the case when we use *effective* numbers of parties instead. Since the effective number of vote-winning parties can take non-integer values, we can define the threshold of representation for any case where $n_{v2} > 1$.

The left-hand panel of Figure 1 shows the thresholds of representation and exclusion where the district magnitude is fixed such that $m = 1$ and where we allow the effective number of vote-winning parties, n_{v2} , to take all possible values between one and ten. By definition, parties are guaranteed to win a seat where they exceed the threshold of exclusion. However,

the threshold of representation, t_R , exceeds the threshold of exclusion, t_X , in all cases where the effective number of vote-winning parties takes any value lower than 2, implying that the point at which any given party first secures a chance to win a seat occurs at a *larger* vote share than the point at which it *must* win a seat.

Clearly, this would make little sense. The thresholds of representation, t_R , and exclusion, t_X , represent the lower and upper bounds on the chances of winning a seat, respectively, so the former must not exceed the latter. Instead, the threshold of representation should *increase* as the effective number of vote-winning parties *decreases* until it reaches the threshold of exclusion, at which point the two should remain equal. Computing this new lower bound is straightforward: as the middle panel of Figure 1 shows, we need only take the minimum of the two thresholds, $t_{\min} = \min(t_R, t_X)$.

So far, we have modelled the probability that any party will win at least one seat in a district (Equation 7) and the appropriate lower ($t_{\min}$) and upper (t_X) bounds on the effective threshold, t . We can now turn to the effective threshold itself. Recall that, by taking the mean of the two thresholds, Lijphart (1994) and Taagepera (2002) allocate each bounding threshold equal weight. However, since we know the district magnitude, m , and the effective number of vote-winning parties, n_{v2} , we can use Equation 7 to compute the probability that any party will win a seat in a district and use this to weight the bounding thresholds instead.

At present, we know only that the effective threshold, t , must lie between $t_{\min}$ and t_X such that $t_{\min} \leq t \leq t_X$. Absent any other information, the simplest assumption is that the chance of winning a seat increases between them in a linear manner. Under this assumption, the probability that a given party wins a given seat at some vote share t is $(t - t_{\min}) / (t_X - t_{\min})$. Thus, if we set this equal to the probability θ from Equation 7, then solve for t , we arrive at a logical model of the effective threshold that weights the two bounding thresholds according to the probability, θ , that a party wins a seat in the district:²

$$t = (1 - \theta) t_{\min} + \theta t_X \quad (8)$$

The right-hand panel of Figure 1 plots this quantity where $m = 1$. As we can see, the effective threshold follows a similar trajectory to the lower bounding threshold, $t_{\min}$. Further, its slope suggests that, for any given value of n_{v2} , relatively little additional success is required for a party to move from contention (the lower bound, $t_{\min}$) to an even chance of winning a seat (the effective threshold, t). After this point, however, the task of winning a seat becomes much more difficult: to make up the remaining difference between an even chance (t) and total certainty (the upper bound, t_X) requires a substantial increase in support.

²I provide corresponding R functions in Appendix A to this paper.

3.2 EMPIRICAL CORROBORATION

To corroborate my logic, I rely on district-level data from the Constituency-Level Elections Archive (Kollman et al. 2025).³ I subset these data in two ways. First, I remove uncontested districts (where thresholds are undefined), elections that used statutory thresholds (which confound estimates of the effective threshold), or elections before the 20th Century. Second, I limit my analysis to elections that use simple electoral systems: first-past-the-post or districted party-list proportional representation (Shugart and Taagepera 2017). This leaves 280,569 unique cases, each representing some party or candidate in one of 46,066 districts at one of 604 elections in one of 111 countries between 1900 and 2024.

I fit a modified two-parameter logistic model to these data. Common in item-response theoretic research (de Ayala 2009), the two-parameter logistic model is useful in this case since it includes a threshold parameter, t , that corresponds to the same point as the effective threshold: the vote share, v , at which a party has a 50/50 chance of winning ($w = 1$) or not winning ($w = 0$) a seat. I modify the model in two ways. First, I use my logical models from Equations 7 and 8 to set the effective threshold, t . Second, I model the slope parameter, β , as a function of the two bounding thresholds, t_x and $t_{\min}$, so that it responds to the possible ranges that these bounds can take, and a discrimination parameter, λ which takes larger values the better able the model is to distinguish between cases where a party did or did not win a seat. The full model is written as follows:⁴

$$\begin{aligned}
 w_i &\sim \text{Bernoulli}(\pi_i) \\
 \text{logit}(\pi_i) &= \beta_i [\text{logit}(v_i) - \text{logit}(t_i)] \\
 \beta_i &= \frac{\lambda}{\text{logit}(t_{x_i}) - \text{logit}(t_{\min_i})} \\
 t_i &= (1 - \theta_i) t_{\min_i} + \theta_i t_{x_i} \\
 \theta_i &= 1 - \left(1 - \frac{1}{n_{v2_i}}\right)^{m_i} \\
 \lambda &\sim \text{Normal}(0, 0.5)
 \end{aligned} \tag{9}$$

I compare my model to three competing alternatives. Two resemble the model above, though set t using Taagepera (1998) and Lijphart's (1994) models instead. However, since both rely only on the district magnitude, m , and not the effective number of vote-winning parties, n_{v2} , it is possible that the latter's inclusion might bias the test in my favour. As such, I also include an additional model that uses a similar logic to Taagepera (1998) and Lijphart (1994),

³In conducting this analysis, I discovered a large number of impossible results in the Constituency-Level Elections Archive. This included cases where there were incorrect candidate or party vote totals, incorrect seat totals, incorrect district magnitudes, results assigned to the wrong electoral district, duplicated rows, and various other errors and omissions. Most appeared to be simple transcription errors and, in total, 446 elections included at least one sort of error. As such, I fixed these problem cases by hand by consulting original election returns and, failing that, contemporaneous sources. I have made the Elections Archive team aware of these issues, included all of the necessary code in the replication package that accompanies this paper, and uploaded the corrected data to a GitHub repository for others to use in the meantime [link anonymised].

⁴Appendix B to this paper provides a step-by-step explanation of this model's logic in greater detail.

Table 1: My combinatorial model outperforms three similar empirical models that rely on Taagepera’s (2002) and Lijphart’s (1994) logical models of the effective threshold or taking the mean of the two bounding thresholds. Judging by the relative discrimination parameters, leave-one-out information criteria, and R^2 values, it better discriminates between parties that do and do not gain representation, makes more precise out-of-sample predictions, and explains more variation in data from the Constituency-Level Elections Archive.

	m	n_{v2}	Discrimination, λ	LOOIC	R^2
Combinatorial	✓	✓	14.07 [13.79, 14.36]	16,038	0.92
Arithmetic Mean	✓	✓	8.61 [8.48, 8.75]	37,040	0.77
Taagepera (1998)	✓	×	4.68 [4.61, 4.74]	52,545	0.72
Lijphart (1994)	✓	×	2.58 [2.54, 2.61]	68,109	0.44
Observations				280,569	
Elections					604
Countries					111

though predicts the effective threshold as the mean of the two bounding thresholds, thereby including n_{v2} , such that $t = 0.5/(m + 1) + 0.5/(mn_{v2})$. I evaluate the models’ performance in two ways. First, I compare the variation that each model explains in the outcome based on their respective R^2 scores. Second, I use approximate leave-one-out cross-validation (Vehtari, Gelman, and Gabry 2017) to show how well each model predicts out-of-sample cases. This produces a ‘leave-one-out information criterion’ (LOOIC) which we interpret like an AIC score (i.e. lower values imply a better fit to the data).

Table 1 presents the resulting estimates and model fit statistics. First, note that my model has a much larger discrimination parameter, λ , than any alternative model, suggesting that it is better able to discriminate between parties that did or did not win a seat. Next, consider the models’ respective LOOIC scores. The competing models based on Taagepera (2002) and Lijphart (1994) have scores of 37,040, 52,545 and 68,109. By contrast, my model has a much smaller score at just 16,038, implying that it makes better out-of-sample predictions. The same is true where we compare the models based on the amount of variation that they explain in the outcome. The competing models have R^2 values of 0.77, 0.72 and 0.44, respectively. My model, however, has an R^2 of 0.92 instead, suggesting that it explains much more variation.

We can also test how well-calibrated each model is by comparing its predictions to real world data. To do so, I draw a bandwidth of 1 to 5 percentage points around each estimate of the effective threshold. For all cases within the bandwidth, I then compute the proportion that win at least one seat in their district. Since the effective threshold represents the point where a party has a 50/50 chance of winning a seat, the resulting score should be 50% if the model is perfectly calibrated. Figure 2 shows the resulting proportions. All three competing models show poor calibration regardless of bandwidth. Indeed, all deviate from perfect calibration by at least 10 percentage points, though this figure is often higher. By contrast, my model shows a much higher degree of calibration, varying between 47% with a bandwidth of 1 percentage point and 49% with a bandwidth of 3 percentage points compared to a perfect score of 50%.

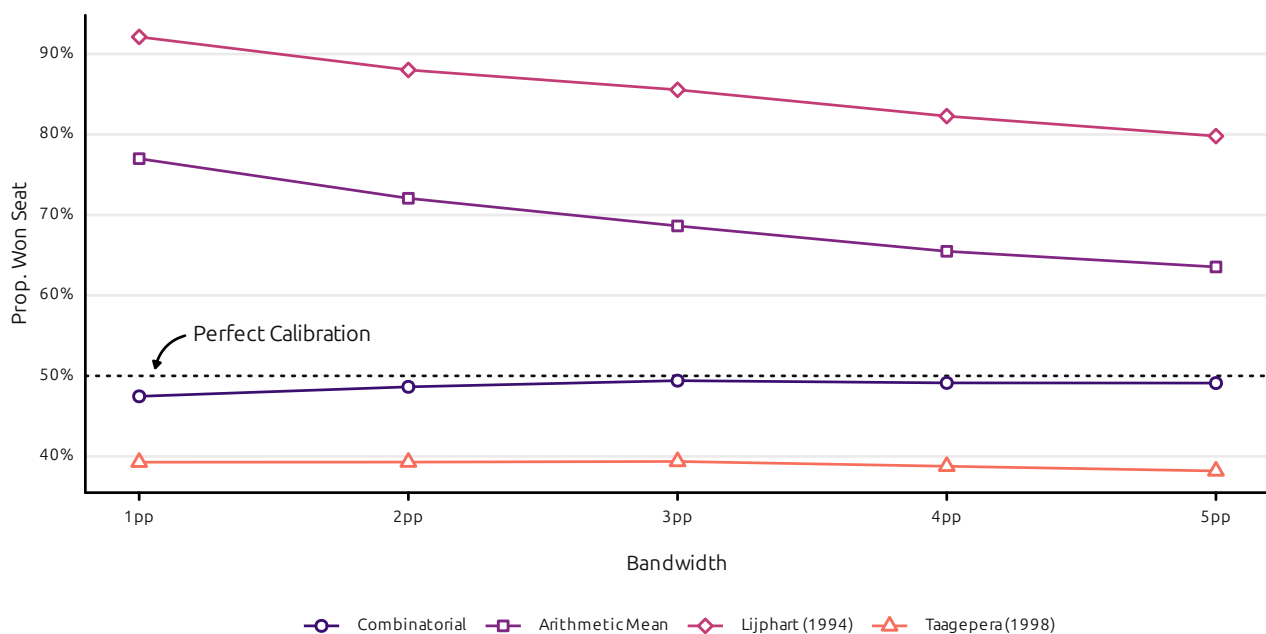


Figure 2: My combinatorial model remains well calibrated across bandwidths, with approximately half of all parties winning at least one seat. By contrast, the three competing models deviate from the 50% benchmark (shown by the dotted line) by at least 10 percentage points, in all cases.

3.3 IMPLICATIONS

We now have compelling evidence that my model out-competes existing alternatives: it better discriminates between parties that do and do not gain representation, it makes more precise out-of-sample predictions, it explains much more variation in the probability that parties do or do not win a seat, and it is much better calibrated to observations from the real world. Next, we must consider its implications. To this end, Figure 3 shows predictions from the model where we let $m = 1, 5, 10$ and $n_{v2} = 1.5, 5, 10$.

Past research suggests that effective thresholds vary with the district magnitude, m (Shugart and Taagepera 2017; Taagepera 2002, 1998; Lijphart 1994). As we can see from Figure 3, my model's predictions are consistent with this basic fact. Effective thresholds are largest where we set $m = 1$ (top row), then fall as we set $m = 5$ (middle row) and then $m = 10$ (bottom row). This suggests one mechanism through which proportional systems produce greater party-system fragmentation: their lower effective thresholds give more parties the chance to gain enough votes to win representation. As such, it also suggests that democratic designers might stem the recent rise in fragmentation (Norris 2024) by using electoral districts with low district magnitudes (Carey and Hix 2011).

Unlike previous research, however, my model also suggests that effective thresholds are endogenous to party system fragmentation. That is to say, two parties with identical vote shares, competing under the same rules, in districts of the same magnitude, can face very different probabilities of representation simply because support for their opponents has a different distribution. As Figure 3 shows, for any given value of m , increasing the effective number of vote-winning parties, n_{v2} , serves to decrease the effective threshold. This cannot

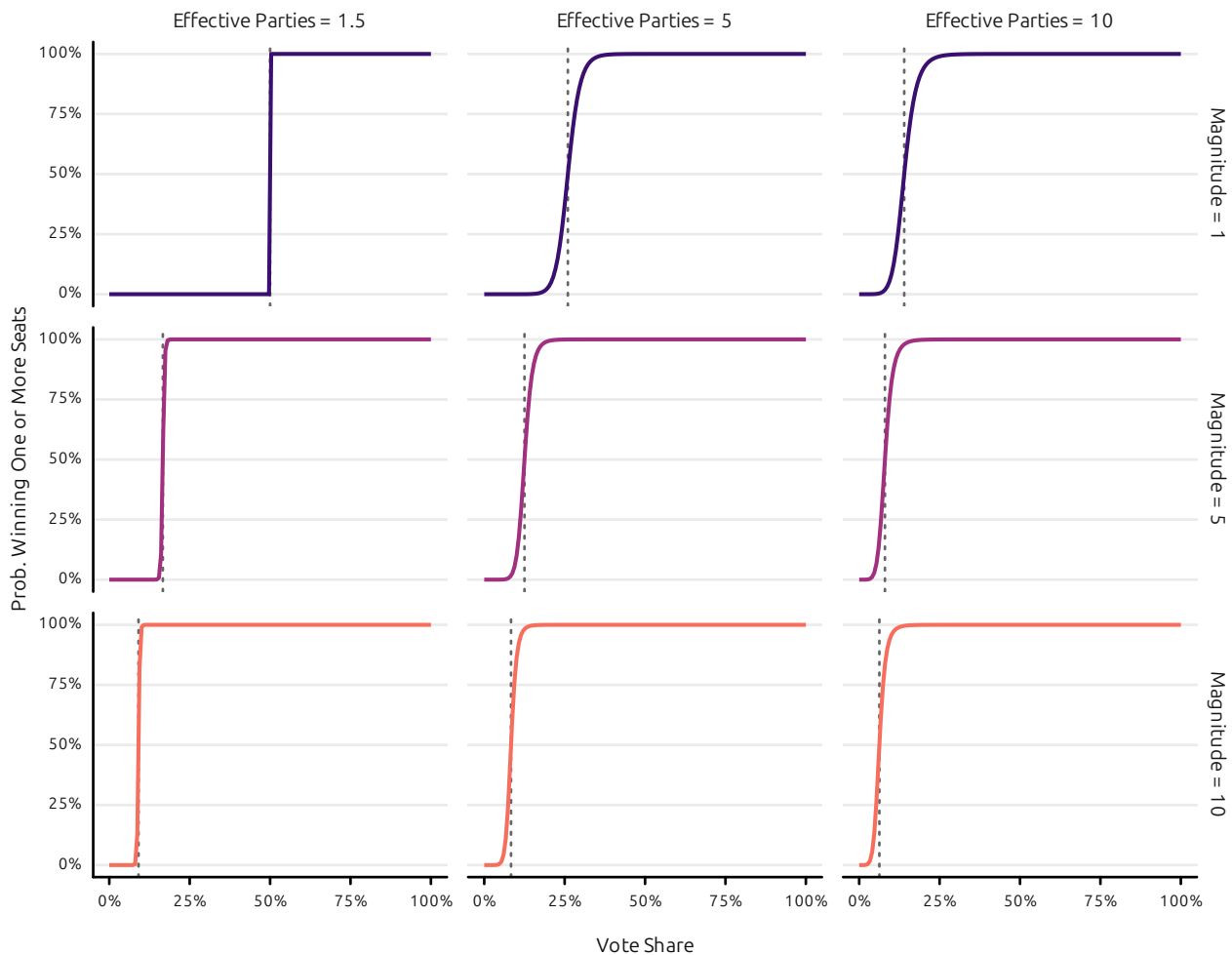


Figure 3: As the district magnitude and the effective number of vote-winning parties *increases*, the effective threshold (here shown as a dotted vertical line) *decreases* in turn. Likewise, where the district magnitude and the effective number of vote-winning parties are small, we are better able to discriminate between parties that did and did not win a seat. Predictions include a point estimate and a 95% credible interval (though the latter is extremely narrow).

be a purely mechanical effect as with the district magnitude. There are many reasons why fragmentation might occur, but one is that voters in proportional systems feel less need to vote strategically than voters in majoritarian ones (Hix, Hortala-Vallve, and Riambau-Armet 2017). In other words, the effect is, in part, psychological (Duverger 1964). If rising fragmentation is a cause for concern, then this finding suggests another benefit of electoral districts with low district magnitudes: by reducing incentives to vote for smaller parties, low magnitude districts raise effective thresholds and reduce fragmentation in the legislature.

This finding has practical implications for party strategy. At a district-level, parties often estimate how many votes they need to win, set a goal around the estimate, and then plan their campaign accordingly (Nielsen 2012, 108; and, for a practical example, see Roberts 2014). However, my model would suggest that party strategists should also consider the district's fragmentation. The 2015 Canadian federal election provides a useful example. Monique Pauzé won in Repentigny on 34.7% of the vote. Yet Gordon Giesbrecht, who also received 34.7% of the

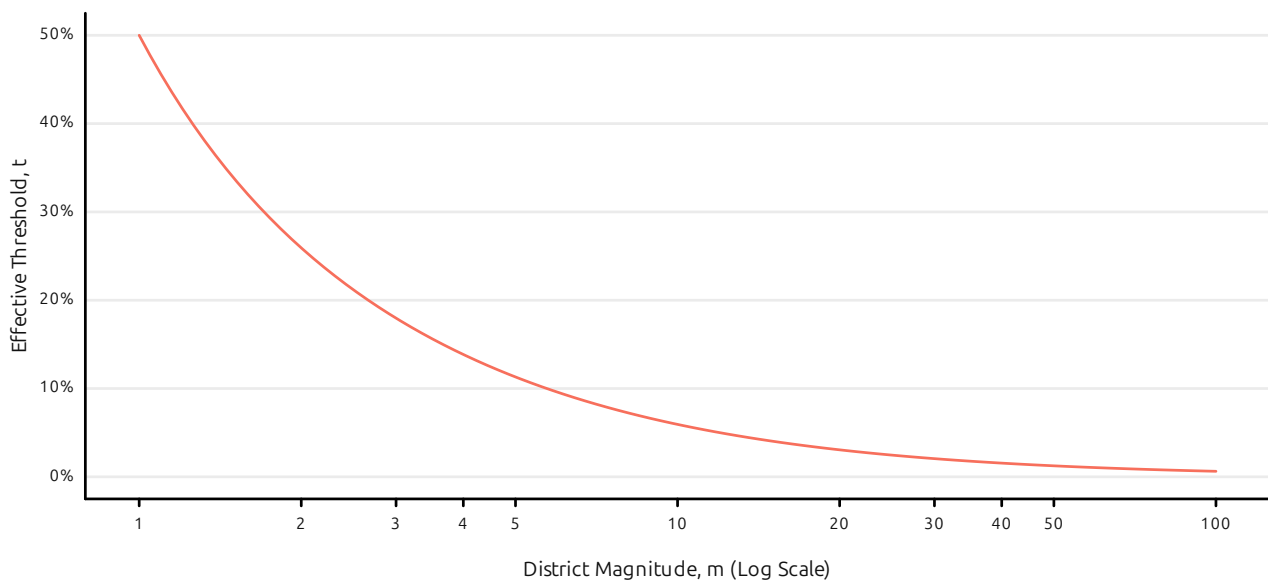


Figure 4: If we model the effective number of vote-winning parties using Cox's (1997) $n_{v2} = m + 1$, we find that the effective threshold declines rapidly as the district magnitude grows.

vote, lost in Winnipeg South. Canadian federal elections use single-member district plurality rules, so all districts have magnitudes of one. But, since the vote was more fragmented in Repentigny ($n_{v2} = 3.83$) than it was in Winnipeg South ($n_{v2} = 2.16$), the effective threshold was lower for Pauzé ($t = 32\%$) than it was for Giesbrecht ($t = 48\%$). Thus, to be maximally effective, party strategists should set vote goals that are specific to the district context and reflect the level of fragmentation that they expect to observe.

In some cases, we will not know the effective number of vote-winning parties in advance. Likewise, it may be useful to know how the effective threshold varies where we combine both the mechanical and psychological effects of changes in the district magnitude. Thankfully, Taagepera-style logical models provide a means to achieve this. Since they are interlocking, with predictions from one model feeding into the next (Taagepera 2008), we can use existing work to model the effective number of vote-winning parties as a function of the district magnitude. In particular, we can use Cox's (1997) model of strategic entry: $n_{v2} = m + 1$. Figure 4 shows that the effective threshold declines quickly as m varies from 1 to 100, falling from 50% where $m = 1$ to 26% where $m = 2$ and 10% where $m = 6$.

This result explains why models using only the district magnitude still perform reasonably well. Substituting $n_{v2} = m + 1$ into Equation 8 and simplifying, we get:

$$t = \frac{1}{m+1} \left[1 - \frac{m-1}{m} \left(\frac{m}{m+1} \right)^m \right] \quad (10)$$

As m tends to infinity, $[m/(m+1)]^m$ tends to e^{-1} and $(m+1)/m$ tends to 1. So, as m grows, the effective threshold becomes equivalent to $(1 - e^{-1})/(m+1) \approx 0.632/(m+1)$. Thus, despite their different motivations, a model that closely resembles Taagepera's (1998), though with a slightly smaller value in the numerator, emerges as a limiting case of my combinatorial model where we replace n_{v2} with Cox's (1997) model, $n_{v2} = m + 1$.

4 CONCLUSION

In this note, I elaborated a logical model of the effective electoral threshold: the vote share at which a given party has an equal chance of winning its first seat in a district. I then used data from the Constituency-Level Elections Archive to compare my model to existing alternatives by Taagepera (1998) and Lijphart (1994). In all cases, my model is better able to discriminate between cases where parties did and did not win a seat, makes more precise out-of-sample predictions, and explains more of the variation present in the data.

Substantively, my model suggests that two factors determine the placement of effective electoral thresholds. First is the district magnitude. The more seats that a district holds, the lower its effective threshold, and the easier it is for any party to win its first seat. This suggests an obvious implication for democratic designers: rather than control fragmentation by enshrining arbitrary thresholds in law, they might instead use electoral districts with small district magnitudes and higher effective thresholds. Second is the effective number of vote-winning parties. As the vote distribution fragments, the district's effective threshold declines, raising the probability that any given party will win a seat. This raises normative and empirical questions for future research. In particular, how fair are majoritarian systems that use single-member plurality rules in light of rising party-system fragmentation? And are there limits on how well we can predict which parties will win seats in fragmented party systems?

My findings advance the Taageperan research agenda of parsimonious and interlocking models of electoral systems (Shugart and Taagepera 2017; Taagepera 2008). Future research in the Taageperan tradition might, therefore, explore how factors that predict the effective number of vote-winning parties (including district magnitude) affect party systems by altering the placement of electoral thresholds. However, my findings also have another potential implication for future research in political science. Though still somewhat stylised, my model provides theoretically-grounded assignment probabilities that political scientists might use as propensity scores to estimate the causal effect of a given party securing representation. For instance, future research might use my model's predictions to test for incumbency effects in PR systems, to estimate the local organisational returns to representation, or to determine whether electing particular parties leads to more local private or public investment.

5 REFERENCES

- Bischoff, Carina S. 2009. "National Level Electoral Thresholds: Problems and Solutions." *Electoral Studies* 28 (2): 232–39. <https://doi.org/10.1016/j.electstud.2008.09.003>.
- Bürkner, Paul-Christian. 2017. "Brms: An R Package for Bayesian Multilevel Models Using Stan." *Journal of Statistical Software* 80 (1): 1–28. <https://doi.org/10.18637/jss.v080.i01>.
- Carey, John M, and Simon Hix. 2011. "The Electoral Sweet Spot: Low-Magnitude Proportional Electoral Systems." *American Journal of Political Science* 55 (2): 383–97. <https://doi.org/10.1111/j.1540-5907.2010.00495.x>.

- Cox, Gary W. 1997. *Making Votes Count: Strategic Coordination in the World's Electoral Systems*. 1st ed. Cambridge University Press. <https://doi.org/10.1017/CBO9781139174954>.
- de Ayala, R J. 2009. *The Theory and Practice of Item Response Theory*. New York, NY: The Guildford Press.
- Dinas, Elias, Pedro Riera, and Nasos Roussias. 2015. "Staying in the First League: Parliamentary Representation and the Electoral Success of Small Parties." *Political Science Research and Methods* 3 (2): 187–204. <https://doi.org/10.1017/psrm.2014.38>.
- Duverger, Maurice, ed. 1964. *Political Parties: Their Organization and Activity in the Modern State*. University Paperbacks 82. London, UK: Methuen.
- Grofman, Bernard. 1975. "A Review of Macro-Election Systems." In *German Political Yearbook (Sozialwissenschaftliches Jahrbuch Fur Politik)*, 4:303–52. Munich, Germany: Verlag.
- Hanretty, Chris. 2022. "Party System Polarization and the Effective Number of Parties." *Electoral Studies* 76 (April): 102459. <https://doi.org/10.1016/j.electstud.2022.102459>.
- Hix, Simon, Rafael Hortala-Vallve, and Guillem Rimbau-Armet. 2017. "The Effects of District Magnitude on Voting Behavior." *The Journal of Politics* 79 (1): 356–61. <https://doi.org/10.1086/688889>.
- Kollman, Ken, Allen Hicken, Daniele Caramani, David Backer, and David Lublin. 2025. "The Elections Archive." Ann Arbor, MI.
- Laakso, Markku, and Rein Taagepera. 1979. "'Effective' Number of Parties: A Measure with Application to West Europe." *Comparative Political Studies* 12 (1): 3–27. <https://doi.org/10.1177/001041407901200101>.
- Lijphart, Arend. 1994. *Electoral Systems and Party Systems: A Study of Twenty-Seven Democracies, 1945–1990*. Oxford, UK: Oxford University Press.
- Lijphart, Arend, and Robert W. Gibberd. 1977. "Thresholds and Payoffs in List Systems of Proportional Representation." *European Journal of Political Research* 5 (3): 219–44. <https://doi.org/10.1111/j.1475-6765.1977.tb01289.x>.
- Nielsen, Rasmus Kleis. 2012. *Ground Wars: Personalized Communication in Political Campaigns*. Princeton: Princeton University Press.
- Norris, Pippa. 2024. "'Things Fall Apart, the Center Cannot Hold': Fractionalized and Polarized Party Systems in Western Democracies." *European Political Science* 23 (4): 546–66. <https://doi.org/10.1057/s41304-023-00467-0>.
- Rae, Douglas, Victor Hanby, and John Loosemore. 1971. "Thresholds of Representation and Thresholds of Exclusion: An Analytic Note on Electoral Systems." *Comparative Political Studies* 3 (4): 479–88. <https://doi.org/10.1177/001041407100300406>.
- Roberts, Marcus. 2014. "Labours Next Majority: A Constituency Guide." London, UK: Fabian Society.
- Rokkan, Stein. 1968. "Elections: Electoral Systems." In *International Encyclopedia of the Social Sciences*, edited by David L. Sills. Vol. 5. New York, NY: Crowell Collier-Macmillan.
- Ruiz-Rufino, Rubén. 2007. "Aggregated Threshold Functions or How to Measure the Performance of an Electoral System." *Electoral Studies* 26 (2): 492–502. <https://doi.org/10.1016/j.electstud.2007.03.001>.

electstud.2006.10.015.

- . 2011. “Characterising Electoral Systems: An Empirical Application of Aggregated Threshold Functions.” *West European Politics* 34 (2): 256–81. <https://doi.org/10.1080/01402382.2011.546570>.
- Shugart, Matthew S., and Rein Taagepera. 2017. *Votes from Seats: Logical Models of Electoral Systems*. Cambridge, UK: Cambridge University Press.
- Taagepera, Rein. 1998. “Effective Magnitude and Effective Threshold.” *Electoral Studies* 17 (4): 393–404. [https://doi.org/10.1016/S0261-3794\(97\)00053-X](https://doi.org/10.1016/S0261-3794(97)00053-X).
- . 2002. “Nationwide Threshold of Representation.” *Electoral Studies* 21 (3): 383–401. [https://doi.org/10.1016/S0261-3794\(00\)00045-7](https://doi.org/10.1016/S0261-3794(00)00045-7).
- . 2008. *Making Social Sciences More Scientific: The Need for Predictive Models*. Oxford, UK: Oxford University Press.
- Taagepera, Rein, and Matthew S. Shugart. 1989. *Seats and Votes: The Effects and Determinants of Electoral Systems*. New Haven, Conn.: Yale University Press.
- Vehtari, Aki, Andrew Gelman, and Jonah Gabry. 2017. “Practical Bayesian Model Evaluation Using Leave-One-Out Cross-Validation and WAIC.” *Statistics and Computing* 27 (5): 1413–32. <https://doi.org/10.1007/s11222-016-9696-4>.

A R FUNCTIONS

```
# Compute the thresholds of representation and exclusion

tR <- function(m, n){ 1 / ( m * n ) }

tX <- function(m){ 1 / ( m + 1 ) }

# Compute the probability of winning at least one seat

prob_rep <- function(m, n){ 1 - ( 1 - 1 / n ) ^ m }

# Compute the effective threshold

effective_threshold <-
  function(m, n){

    ( 1 - prob_rep(m, n) ) * pmin( tR(m, n), tX(m) ) + prob_rep(m, n) * tX(m)

  }
```

B MODEL DERIVATION

In the main body of the paper, I present a non-linear Bayesian model that I fit to data from the Constituency-Level Elections Archive (Kollman et al. 2025). In this appendix, I derive the model step-by-step, setting out the decisions that informed it. We begin with the outcome variable: the probability that a given party did ($w = 1$) or did not ($w = 0$) win at least one seat in its district. Since this variable is binary, we can assume that it is Bernoulli distributed according to some probability, π :

$$w_i \sim \text{Bernoulli}(\pi_i) \quad (\text{A1})$$

I begin by modelling the log odds of π using a simple two-parameter logistic model. This model has a slope parameter, β , that determines how well the model discriminates between cases that did and did not win a seat, and a threshold parameter, t , that is equivalent to the effective threshold, according to each party's vote share, v_i :

$$\begin{aligned} w_i &\sim \text{Bernoulli}(\pi_i) \\ \text{logit}(\pi_i) &= \beta(v_i - t) \end{aligned} \quad (\text{A2})$$

At present, the model can make impossible predictions. For example, where the threshold parameter, t , is low—which might occur in districts with large magnitudes and many vote-winning parties—the model could predict that parties have a chance of winning a seat even where their vote share is 0%. To account for this, we can put each party's vote share, v , and the model's estimate of the effective threshold, t , on the logit scale by taking their log odds. This forces the model to interpret vote shares of 0% and 100% as implying probabilities of winning at least one seat of 0% and 100%, respectively:

$$\begin{aligned} w_i &\sim \text{Bernoulli}(\pi_i) \\ \text{logit}(\pi_i) &= \beta [\text{logit}(v_i) - \text{logit}(t)] \end{aligned} \tag{A3}$$

Another limitation that the model faces at present is that it has only a single effective threshold no matter the district magnitude, m , or the effective number of vote-winning parties, n_{v2} . However, we know what value this threshold should take due to Equation 8 in the main paper. As such, we do not need to estimate the effective threshold. Instead, we can substitute in the necessary equations as follows:

$$\begin{aligned} w_i &\sim \text{Bernoulli}(\pi_i) \\ \text{logit}(\pi_i) &= \beta [\text{logit}(v_i) - \text{logit}(t_i)] \\ t_i &= (1 - \theta_i) t_{\min_i} + \theta_i t_{x_i} \\ \theta_i &= 1 - \left(1 - \frac{1}{n_{v2_i}}\right)^{m_i} \end{aligned} \tag{A4}$$

The one final change that we must make concerns the discrimination parameter, β . We know that this parameter must vary depending on the district magnitude, m , and the effective number of vote-winning parties, n_{v2} . After all, the bounding thresholds, t_R and t_X , differ from case to case as a function of these variables, so the value that β takes must change to accommodate the distance between them. The *smallest* distance between the bounding thresholds occurs where $t_R \leq t_X$ and the two thresholds converge on a single value (e.g. where $m = 1$ and $n_{v2} = 2$, $t_{\min} = t_X = 0.5$). The *largest* distance, instead, occurs where $m = 1$ and n_{v2} tends to ∞ , since the threshold of exclusion equals its maximum value of 50%, but the threshold of representation converges on 0%, producing a 50 percentage point distance.

This suggests that we should let the discrimination parameter, β , vary as a function of the distance between lower, $t_{\min}$, and upper, t_X , bounding thresholds. To do so, recall that, where $t_R \geq t_X$, $t_{\min}$ and t_X are both equal. Thus, the slope here *must* be infinite to allow for a discontinuous jump in the probability of winning a seat at the electoral threshold. Imagine some new parameter, λ , that represents this 100 percentage point change, though measured in units of logits. If we set $v = t_X$ and $t = t_{\min}$ to reflect the limiting cases where $t_R \geq t_X$, and express λ in terms of β and the two bounding thresholds such that $\lambda = \beta [\text{logit}(t_X) - \text{logit}(t_{\min})]$, we can rearrange to solve for β :⁵

⁵In all cases where $t_R \leq t_X$, a division by zero occurs. This prevents the model from fitting, since it evaluates to a value of infinity (Inf). Due to the model's functional form, these cases are all *perfectly* identified. In effect, they

$$\beta_i = \frac{\lambda}{\text{logit}(t_{x_i}) - \text{logit}(t_{\min_i})} \quad (\text{A5})$$

Finally, we can substitute this into Equations A4 above and include a weakly-informative prior on λ to get our final empirical model:

$$\begin{aligned} w_i &\sim \text{Bernoulli}(\pi_i) \\ \text{logit}(\pi_i) &= \beta_i [\text{logit}(v_i) - \text{logit}(t_i)] \\ \beta_i &= \frac{\lambda}{\text{logit}(t_{x_i}) - \text{logit}(t_{\min_i})} \\ t_i &= (1 - \theta_i) t_{\min_i} + \theta_i t_{x_i} \\ \theta_i &= 1 - \left(1 - \frac{1}{n_{v2_i}}\right)^{m_i} \\ \lambda &\sim \text{Normal}(0, 0.5) \end{aligned} \quad (\text{A6})$$

have an infinitely large amount of statistical power since we know the exact value that the effective threshold must take: $t = t_x = t_{\min}$. As such, I drop any cases where $t_R \geq t_x$ from my data so that the model is able to sample reliably.